INNER PRODUCT SPACES

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Throughout these notes, we study vector spaces over a scalar field $\mathbb{F}$, which is either $\mathbb{R}$ or $\mathbb{C}$. The set of positive integers is denoted by $\mathbb{N}$, and its elements are i, j, k, l, m, n, p. For a nonempty finite set A, the number of elements in A is denoted by $\#A \in \mathbb{N}$, with $\#\emptyset = 0$.

Vector spaces and sets of vectors are denoted by capital calligraphic letters, such as $\mathscr{V}, \mathscr{X}, \mathscr{A}$, etc. Vectors in abstract vector spaces are denoted by lowercase Latin letters, such as u, v, x, y, etc. Linear operators are denoted by uppercase Latin letters, such as S, T, etc. Scalars are represented by lowercase Greek letters, such as α, β, etc.

Vectors in Euclidean spaces $\mathbb{F}^n$ are denoted by boldface lowercase letters, such as $\mathbf{a}, \mathbf{b}$, etc. Matrices with entries in $\mathbb{F}$ are denoted by uppercase Latin letters in a sans-serif font, such as $\mathsf{A}, \mathsf{B}, \mathsf{M}$, etc. The $n \times n$ identity matrix is denoted by I_n, while $\mathsf{0}$ represents a zero matrix whose size is implied by the context. The transpose of a matrix M is denoted by $\mathsf{M}^\top$.

Pay attention to exceptions to these conventions. If you notice significant deviations, please let me know.

1. Inner Product Spaces

By i we denote the imaginary unit in $\mathbb{C}$. For a complex number α by $\operatorname{Re}(\alpha)$ we denote the real part of α and by $\operatorname{Im}(\alpha)$ we denote the imaginary part of α. We have $\alpha = \operatorname{Re}(\alpha) + \mathrm{i}\operatorname{Im}(\alpha)$. By $\overline{\alpha}$ we denote the conjugate of of α. We have $\overline{\alpha} = \operatorname{Re}(\alpha) - \mathrm{i}\operatorname{Im}(\alpha)$. By $|\alpha|$ we denote the modulus of α. We have

$$|\alpha| = \sqrt{\alpha\overline{\alpha}} = \sqrt{(\operatorname{Re}\alpha)^2 + (\operatorname{Im}\alpha)^2}.$$

In this section $\mathbb{F}$ stands for either $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.1. Let $\mathscr{V}$ be a vector space over $\mathbb{F}$. If the following five conditions are satisfied

IPE. $\langle \cdot , \cdot \rangle : \mathscr{V} \times \mathscr{V} \to \mathbb{F}$ is a function,
IPL. $\forall u, v, w \in \mathscr{V} \ \ \forall \alpha, \beta \in \mathbb{F} \ \ \langle \alpha u + \beta v, w\rangle = \alpha\langle u, w\rangle + \beta\langle v, w\rangle$,
IPC. $\forall u, v \in \mathscr{V} \quad \langle v, u\rangle = \overline{\langle u, v\rangle}$,
IPN. $\forall v \in \mathscr{V} \quad \langle v, v\rangle \geq 0$,
IPD. $\forall v \in \mathscr{V} \quad \langle v, v\rangle = 0$ implies $v = 0_{\mathscr{V}}$,

then the function $\langle \cdot , \cdot \rangle : \mathscr{V} \times \mathscr{V} \to \mathbb{F}$ is called a **positive definite inner product** on $\mathscr{V}$. The ordered pair $(\mathscr{V}, \langle \cdot , \cdot \rangle)$ of a vector space over $\mathbb{F}$ and a positive definite inner product is called an **inner product space** over $\mathbb{F}$. $\Diamond$

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In these notes an inner product space means a vector space with a **positive definite** inner product.

In the following proposition we establish basic algebra on a vector space with a positive definite inner product.

Proposition 1.2. *Let $(\mathscr{V}, \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{F}$. The following statements hold.*

(a) $\forall v \in \mathscr{V}$ *we have* $\langle 0_{\mathscr{V}}, v\rangle = \langle v, 0_{\mathscr{V}}\rangle = 0$.
(b) $\forall u, v, w \in \mathscr{V}$ $\forall \alpha, \beta \in \mathbb{F}$ *we have* $\langle u, \alpha v + \beta w\rangle = \overline{\alpha}\langle u, v\rangle + \overline{\beta}\langle u, w\rangle$.
(c) *For all* $m, n \in \mathbb{N}$ *and all* $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \in \mathbb{F}$ *and all* $u_1, \ldots, u_m$, $v_1, \ldots, v_n \in \mathscr{V}$ *we have*

$$\left\langle \sum_{j=1}^{m} \alpha_j u_j, \sum_{k=1}^{n} \beta_k v_k \right\rangle = \sum_{j=1}^{m}\sum_{k=1}^{n} \alpha_j \overline{\beta}_k \langle u_j, v_k\rangle.$$

Proof. □

Definition 1.3. Let $(\mathscr{V}, \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{F}$. Vectors $u, v \in \mathscr{V}$ are said to be **orthogonal** if $\langle u, v\rangle = 0$. The notation for orthogonal vectors is $u \perp v$.

A set of nonzero vectors $\mathscr{A} \subset \mathscr{V} \setminus \{0_{\mathscr{V}}\}$ is said to form an **orthogonal system** in $(\mathscr{V}, \langle \cdot, \cdot \rangle)$ if for all $u, v \in \mathscr{A}$ we have $\langle u, v\rangle = 0$ whenever $u \neq v$ and for all $v \in \mathscr{A}$ we have $\langle v, v\rangle > 0$. An orthogonal system $\mathscr{A}$ is called an **orthonormal system** if for all $v \in \mathscr{A}$ we have $\langle v, v\rangle = 1$. ♢

Theorem 1.4 (Pythagorean Theorem). *Let $(\mathscr{V}, \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{F}$ and let $n \in \mathbb{N}$. If $\{v_1, \cdots, v_n\} \subset \mathscr{V}$ is an orthogonal system in $(\mathscr{V}, \langle \cdot, \cdot \rangle)$, then*

$$\left\langle \sum_{j=1}^{n} v_j, \sum_{k=1}^{n} v_k \right\rangle = \sum_{j=1}^{n} \langle v_j, v_j\rangle.$$

Proof. Assume that $\{v_1, \cdots, v_n\} \subset \mathscr{V}$ is orthogonal system in $(\mathscr{V}, \langle \cdot, \cdot \rangle)$. That is, assume that for all $j, k \in \{1, \ldots, n\}$ we have $\langle v_j, v_k\rangle = 0$ whenever $j \neq k$ and $\langle v_k, v_k\rangle > 0$. Then we have

$$\begin{aligned}
\left\langle \sum_{j=1}^{n} v_j, \sum_{k=1}^{n} v_k \right\rangle &= \sum_{j=1}^{n} \left\langle v_j, \sum_{k=1}^{n} v_k \right\rangle \\
&= \sum_{j=1}^{n} \left\langle v_j, v_j + \sum_{\substack{k=1\\k\neq j}}^{n} v_k \right\rangle \\
&= \sum_{j=1}^{n} \left(\langle v_j, v_j\rangle + \left\langle v_j, \sum_{\substack{k=1\\k\neq j}}^{n} v_k \right\rangle \right)
\end{aligned}$$

$$= \sum_{j=1}^{n} \langle v_j, v_j \rangle + \sum_{j=1}^{n} \left\langle v_j, \sum_{\substack{k=1\\k\neq j}}^{n} v_k \right\rangle$$

$$= \sum_{j=1}^{n} \langle v_j, v_j \rangle + \sum_{j=1}^{n} \left(\sum_{\substack{k=1\\k\neq j}}^{n} \langle v_j, v_k \rangle \right)$$

$$= \sum_{j=1}^{n} \langle v_j, v_j \rangle.$$

The first equality follows from the additivity property in the first variable of the inner product (a special case of Proposition 1.2(c)). The second equality follows from the commutativity of addition in the vector space. The third equality follows from the additivity property in the second variable of the inner product (a special case of Proposition 1.2(b)). The fourth equality follows from the commutativity of addition in $\mathbb{C}$. The fifth equality follows from the additivity property in the second variable of the inner product (a special case of Proposition 1.2(c)). The sixth equality follows from the assumption for all $j, k \in \{1, \dots, n\}$ we have $\langle v_j, v_k \rangle = 0$ whenever $j \neq k$. $\square$

Remark 1.5. One could have stated that the Pythagorean theorem follows from Proposition 1.2(c), but that would have obscured the details of the reasoning. I also wanted to emphasize that the only property of the inner product that is used in the proof is the additivity property of the inner product. $\Diamond$

Theorem 1.6 (Cauchy-Bunyakovsky-Schwartz Inequality)**.** *Let* $(\mathscr{V}, \langle \cdot, \cdot \rangle)$ *be an inner product space over* $\mathbb{F}$*. Then*

$$\forall u, v \in \mathscr{V} \qquad \bigl|\langle u, v\rangle\bigr|^2 \leq \langle u, u\rangle \langle v, v\rangle. \tag{1.1}$$

The equality occurs in (1.1) *if and only if* u *and* v *are linearly dependent.*

Proof. **Proof 1.** Let $u, v \in \mathscr{V}$ be arbitrary. **Case 1.** Assume $v = 0_{\mathscr{V}}$. By Proposition 1.2(a) we have $\langle u, v\rangle = 0$ and $\langle v, v\rangle = 0$. Therefore the Cauchy-Bunyakovsky-Schwartz Inequality holds as an equality.

Case 2. Assume $v \neq 0_{\mathscr{V}}$. Then $\langle v, v\rangle > 0$. Consider the vector

$$u - \frac{\langle u, v\rangle}{\langle v, v\rangle} v.$$

Then by Proposition 1.2(c), and using the fact that $\langle v, v\rangle > 0$ we have

$$\left\langle u - \frac{\langle u, v\rangle}{\langle v, v\rangle} v, u - \frac{\langle u, v\rangle}{\langle v, v\rangle} v \right\rangle$$

$$= \langle u, u\rangle - \frac{\overline{\langle u, v\rangle}}{\langle v, v\rangle}\langle u, v\rangle - \frac{\langle u, v\rangle}{\langle v, v\rangle}\langle v, u\rangle + \frac{\langle u, v\rangle\overline{\langle u, v\rangle}}{\langle v, v\rangle^2}\langle v, v\rangle$$

$$= \langle u, u\rangle - \frac{|\langle u, v\rangle|^2}{\langle v, v\rangle}.$$

Since $\langle v, v\rangle > 0$, the established equality is equivalent to

$$\langle v, v\rangle \left\langle u - \frac{\langle u, v\rangle}{\langle v, v\rangle} v, u - \frac{\langle u, v\rangle}{\langle v, v\rangle} v \right\rangle = \langle u, u\rangle\langle v, v\rangle - |\langle u, v\rangle|^2. \tag{1.2}$$

Since the left-hand side in (1.2) is nonnegative, we have that

$$0 \leq \langle u, u\rangle\langle v, v\rangle - |\langle u, v\rangle|^2,$$

which is equivalent to the Cauchy-Bunyakovsky-Schwartz Inequality.

The case of equality. Assume that u and v are linearly dependent. If $v = 0_{\mathscr{V}}$, then equality in the Cauchy-Bunyakovsky-Schwartz Inequality holds. If $v \neq 0_{\mathscr{V}}$, then there exists $\alpha \in \mathbb{F}$ such that $u = \alpha v$. Then

$$|\langle u, v\rangle|^2 = |\alpha|^2 \langle v, v\rangle^2 \quad \text{and} \quad \langle u, u\rangle\langle v, v\rangle = |\alpha|^2\langle v, v\rangle^2.$$

Thus the equality in the Cauchy-Bunyakovsky-Schwartz Inequality holds.

To prove the converse, assume that the equality holds in the Cauchy-Bunyakovsky-Schwartz Inequality. Then both sides of the equality in (1.2) equal to 0. If $v = 0_{\mathscr{V}}$, then u and v are linearly dependent. If $v \neq 0_{\mathscr{V}}$, then $\langle v, v\rangle > 0$. Consequently, (1.2), by IPD in Definition 1.1, implies that

$$u - \frac{\langle u, v\rangle}{\langle v, v\rangle} v = 0_{\mathscr{V}}.$$

This proves that u and v are linearly dependent. **End of Proof 1.**

Proof 2. Let $u, v \in \mathscr{V}$ be arbitrary. **Case 2.** Assume $v \neq 0_{\mathscr{V}}$. As is calculated in Figure 1, the vectors on the right hand side of the following decomposition of u,

$$u = \frac{\langle u, v\rangle}{\langle v, v\rangle} v + \left(u - \frac{\langle u, v\rangle}{\langle v, v\rangle} v \right)$$

are orthogonal. With the notation

$$w = u - \frac{\langle u, v\rangle}{\langle v, v\rangle} v,$$

an application of Pythagorean Theorem yields

$$\langle u, u\rangle = \left\langle \frac{\langle u, v\rangle}{\langle v, v\rangle} v, \frac{\langle u, v\rangle}{\langle v, v\rangle} v \right\rangle + \langle w, w\rangle. \tag{1.3}$$

Using algebra of inner product, (1.3) is equivalent to:

$$\langle u, u\rangle = \frac{|\langle u, v\rangle|^2}{\langle v, v\rangle} + \langle w, w\rangle. \tag{1.4}$$

Since $\langle w, w\rangle \geq 0$, we have that (1.4) implies

$$\langle u, u\rangle \geq \frac{|\langle u, v\rangle|^2}{\langle v, v\rangle}. \tag{1.5}$$

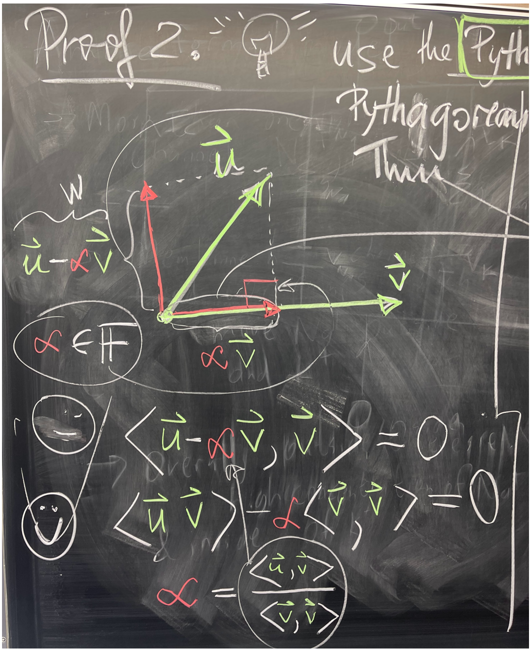

FIGURE 1. Orthogonal decomposition of u along v and $\perp v$.

Multiplying the last inequality by $\langle v, v\rangle > 0$, we obtain the Cauchy-Bunyakovsky-Schwarz Inequality:

$$\langle u, u\rangle\langle v, v\rangle \geq |\langle u, v\rangle|^2. \tag{1.6}$$

The case of equality. We prove only the second part. Assume that the equality holds in (1.6). Then the equality holds in (1.5). Comparing (1.5) and (1.4) we deduce that $\langle w, w\rangle = 0$. Now, IPD in Definition 1.1 implies $w = 0_{\mathscr{V}}$. By the definition of w this proves that u and v are linearly dependent. $\square$

Definition 1.7. Let $\mathscr{V}$ be a vector space over $\mathbb{F}$, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. If the following five conditions are satisfied:

NE. $\|\cdot\| : \mathscr{V} \to \mathbb{F}$ is a function,
NS. $\forall v \in \mathscr{V} \ \ \forall \alpha \in \mathbb{F} \qquad \|\alpha v\| = |\alpha|\|v\|$,
NT. $\forall u, v \in \mathscr{V} \qquad \|u + v\| \leq \|u\| + \|v\|$,
NN. $\forall v \in \mathscr{V} \qquad \|v\| \geq 0$,
ND. $\forall v \in \mathscr{V} \qquad \|v\| = 0$ implies $v = 0_{\mathscr{V}}$,

then the function $\|\cdot\| : \mathscr{V} \to \mathbb{F}$ is called a **norm** on $\mathscr{V}$.

A normed space over a field $\mathbb{F}$ is an ordered pair $(\mathscr{V}, \|\cdot\|)$, where $\mathscr{V}$ is a vector space over $\mathbb{F}$ and $\|\cdot\|$ is a norm on $\mathscr{V}$.

In a normed space $(\mathscr{V}, \|\cdot\|)$, the distance between any two vectors $u, v \in \mathscr{V}$ is defined as:

$$\mathrm{dist}(u, v) = \|u - v\|.$$

$\lozenge$

An inner product gives rise to a norm.

Theorem 1.8. *Let $(\mathscr{V}, \langle \cdot\, , \cdot \rangle)$ be an inner product space over $\mathbb{F}$. The function $\|\cdot\| : \mathscr{V} \to \mathbb{F}$ defined by*

$$\forall v \in \mathscr{V} \qquad \|v\| = \sqrt{\langle v, v\rangle} \tag{1.7}$$

is a norm on $\mathscr{V}$.

Proof. The formula in (1.7) defines a function on $\mathscr{V}$ since it represents a composition of two functions. The first function is $v \mapsto \langle v, v\rangle$ defined on $\mathscr{V}$ with the values in the set of nonnegative real numbers, see IPN in Definition 1.1. The second function is the real square root function.

To prove NS, let $v \in \mathscr{V}$ and $\alpha \in \mathbb{F}$ be arbitrary.

To prove NT, let $u, v \in \mathscr{V}$ be arbitrary.

The property NN follows from the definition of the real square root function.

To prove ND, let $v \in \mathscr{V}$ be arbitrary. □

Remark 1.9. The Cauchy-Bunyakovsky-Schwartz Inequality can be restated using the norm defined in Theorem 1.8 as follows. Let $\mathscr{V}$ be a vector space over $\mathbb{F}$ with a positive definite inner product $\langle \cdot\, , \cdot \rangle$. Then

$$\forall u, v \in \mathscr{V} \qquad \big|\langle u, v\rangle\big| \leq \|u\|\,\|v\|. \tag{1.8}$$

♢

Definition 1.10. Let $\big(\mathscr{V}, \langle \cdot\, , \cdot \rangle\big)$ be an inner product space over $\mathbb{F}$ and $n \in \mathbb{N}$. A set of nonzero vectors

$$\{u_1, \ldots, u_n\} \subset \mathscr{V} \setminus \{0_{\mathscr{V}}\}$$

is said to be an **orthogonal system of vectors** in $\big(\mathscr{V}, \langle \cdot\, , \cdot \rangle\big)$ (or, briefly **orthogonal system**) if the following implication holds:

$$\forall j, k \in \{1, \ldots, n\} \quad j \neq k \quad \Rightarrow \quad \langle u_j, u_k\rangle = 0.$$

♢

The following theorem collects important properties of orthogonal systems of vectors.

Theorem 1.11. *Let $\big(\mathscr{V}, \langle \cdot\, , \cdot \rangle\big)$ be an inner product space over $\mathbb{F}$. Let $n \in \mathbb{N}$ and let $\{u_1, \ldots, u_n\} \subset \mathscr{V}$ be an orthogonal system in $(\mathscr{V}, \langle \cdot\, , \cdot \rangle)$, and set $\mathscr{U} = \operatorname{span}\{u_1, \ldots, u_n\}$. The following statements hold.*

(a) *For all $u \in \mathscr{U}$ we have*

$$u = \sum_{j=1}^{n} \alpha_j u_j \quad \Rightarrow \quad \forall j \in \{1, \ldots, n\} \;\; \alpha_j = \frac{\langle u, u_j\rangle}{\langle u_j, u_j\rangle}.$$

In particular, an orthogonal system is linearly independent.

(b) *For every* $v \in \mathscr{V}$ *we have*

$$v - \sum_{j=1}^{n} \frac{\langle v, u_j \rangle}{\langle u_j, u_j \rangle} u_j \quad \perp \quad \mathscr{U}.$$

(c) *For every* $v \in \mathscr{V}$ *Bessel's inequality holds*

$$\sum_{j=1}^{n} \frac{|\langle v, u_j \rangle|^2}{\langle u_j, u_j \rangle} \leq \langle v, v \rangle = \|v\|^2.$$

The equality holds in Bessel's inequality if and only if $v \in \mathscr{U}$.

Proof. To prove (a), let $u = \sum_{j=1}^{n} \alpha_j u_j$, let $k \in \{1, \ldots, n\}$ be arbitrary, and calculate the inner product with u_k for both sides of the equality. Then, using the linearity of the inner product in the first variable and the fact that $\langle u_j, u_k \rangle = 0$ whenever $j \neq k$ we obtain $\langle u, u_k \rangle = \sum_{j=1}^{n} \alpha_j \langle u_j, u_k \rangle = \alpha_k \langle u_k, u_k \rangle$. Since $\langle u_k, u_k \rangle > 0$, we have $\alpha_k = \dfrac{\langle u, u_k \rangle}{\langle u_k, u_k \rangle}$.

To prove (b), let $v \in \mathscr{V}$ be arbitrary. Let let $k \in \{1, \ldots, n\}$ be arbitrary, and calculate the inner product

$$\begin{aligned}
\left\langle v - \sum_{j=1}^{n} \frac{\langle v, u_j \rangle}{\langle u_j, u_j \rangle} u_j, u_k \right\rangle &= \langle v, u_k \rangle - \sum_{j=1}^{n} \frac{\langle v, u_j \rangle}{\langle u_j, u_j \rangle} \langle u_j, u_k \rangle \\
&= \langle v, u_k \rangle - \langle v, u_k \rangle \\
&= 0.
\end{aligned}$$

Since $k \in \{1, \ldots, n\}$ was arbitrary, replacing u_k with an arbitrary vector in $\mathscr{U}$ also leads to the zero inner product.

To prove (c) we observe that the $n+1$ vectors on the right side in the equality

$$v = \left(v - \sum_{j=1}^{n} \frac{\langle v, u_j \rangle}{\langle u_j, u_j \rangle} u_j \right) + \sum_{j=1}^{n} \frac{\langle v, u_j \rangle}{\langle u_j, u_j \rangle} u_j$$

are mutually orthogonal and apply the Pythagorean Theorem to obtain

$$\|v\|^2 = \left\| v - \sum_{j=1}^{n} \frac{\langle v, u_j \rangle}{\langle u_j, u_j \rangle} u_j \right\|^2 + \sum_{j=1}^{n} \frac{|\langle v, u_j \rangle|^2}{\langle u_j, u_j \rangle}.$$

Bessel's inequality and the characterization of the equality follow from the preceding equality. □

Remark 1.12. The formulas that appear in the preceding theorem are probably the most important formulas in inner product spaces. My nickname for the content in (a) is "*Easy Linear Combinations*," since (a) shows that the coefficients of a linear combination of an orthogonal system are given by clear and important-to-remember formulas. The vector

$$\sum_{j=1}^{n} \frac{\langle v, u_j \rangle}{\langle u_j, u_j \rangle} u_j$$

in (b) is called the **orthogonal projection of v onto $\mathscr{U}$**. For more about the orthogonal projections, see paragraphs after Corollary 1.19. My nickname for the content in (b) is "*Easy Orthogonal Projection*," since (b) shows that the coefficients of the orthogonal projection onto a span of an orthogonal system are given by clear and important-to-remember formulas. **Bessel's inequality** needs no nickname; it is one of the key tools in proving convergence of Fourier series. $\Diamond$

Theorem 1.13 (The Gram-Schmidt orthogonalization)**.** *Let $(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle)$ be an inner product space over $\mathbb{F}$. Let $n \in \mathbb{N}$ and let $v_1, \ldots, v_n$ be linearly independent vectors in $\mathscr{V}$. Let the vectors $u_1, \ldots, u_n$ be defined recursively by*

$$\begin{aligned} u_1 &= v_1, \\ u_{k+1} &= v_{k+1} - \sum_{j=1}^{k} \frac{\langle v_{k+1}, u_j \rangle}{\langle u_j, u_j \rangle} u_j, \quad k \in \{1, \ldots, n-1\}. \end{aligned}$$

Then the vectors $u_1, \ldots, u_n$ form an orthogonal system for which the following set equalities hold

$$\forall k \in \{1, \ldots, n\} \quad \operatorname{span}\{u_1, \ldots, u_k\} = \operatorname{span}\{v_1, \ldots, v_k\}.$$

Proof. We will prove by Mathematical Induction the following statement: For all $k \in \{1, \ldots, n\}$ we have:

(a) $\langle u_k, u_k \rangle > 0$ and $\langle u_j, u_k \rangle = 0$ whenever $j \in \{1, \ldots, k-1\}$;
(b) vectors $u_1, \ldots, u_k$ are linearly independent;
(c) $\operatorname{span}\{u_1, \ldots, u_k\} = \operatorname{span}\{v_1, \ldots, v_k\}$.

For $k = 1$ statements (a), (b) and (c) are clearly true. Let $m \in \{1, \ldots, n-1\}$ and assume that statements (a), (b) and (c) are true for all $k \in \{1, \ldots, m\}$.

Next we will prove that statements (a), (b) and (c) are true for $k = m+1$. Recall the definition of u_{m+1}:

$$u_{m+1} = v_{m+1} - \sum_{j=1}^{m} \frac{\langle v_{m+1}, u_j \rangle}{\langle u_j, u_j \rangle} u_j.$$

By the Inductive Hypothesis we have $\operatorname{span}\{u_1, \ldots, u_m\} = \operatorname{span}\{v_1, \ldots, v_m\}$. Since $v_1 \ldots, v_{m+1}$ are linearly independent, $v_{m+1} \notin \operatorname{span}\{u_1, \ldots, u_m\}$. Therefore, $u_{m+1} \neq 0_{\mathscr{V}}$. That is, $\langle u_{m+1}, u_{m+1} \rangle > 0$. Let $k \in \{1, \ldots, m\}$ be arbitrary. Then by the Inductive Hypothesis we have that $\langle u_j, u_k \rangle = 0$ whenever $j \in \{1, \ldots, m\}$ and $j \neq k$. Therefore,

$$\langle u_{m+1}, u_k \rangle = \langle v_{m+1}, u_k \rangle - \sum_{j=1}^{m} \frac{\langle v_{m+1}, u_j \rangle}{\langle u_j, u_j \rangle} \langle u_j, u_k \rangle$$

$$= \langle v_{m+1}, u_k \rangle - \langle v_{m+1}, u_k \rangle$$
$$= 0.$$

This proves claim (a). To prove claim (b) notice that by the Inductive Hypothesis $u_1, \dots, u_m$ are linearly independent and $u_{m+1} \notin \operatorname{span}\{u_1, \dots, u_m\}$ since $v_{m+1} \notin \operatorname{span}\{u_1, \dots, u_m\}$. To prove claim (c) notice that the definition of u_{m+1} implies $u_{m+1} \in \operatorname{span}\{v_1, \dots, v_{m+1}\}$. Since by the inductive hypothesis $\operatorname{span}\{u_1, \dots, u_m\} = \operatorname{span}\{v_1, \dots, v_m\}$, we have $\operatorname{span}\{u_1, \dots, u_{m+1}\} \subseteq \operatorname{span}\{v_1, \dots, v_{m+1}\}$. The converse inclusion follows from the fact that $v_{m+1} \in \operatorname{span}\{u_1, \dots, u_{m+1}\}$.

The claim of the theorem follows from the claim that has been proven. □

The following two statements are immediate consequences of the Gram-Schmidt orthogonalization algorithm.

Corollary 1.14. *If $(\mathscr{V}, \langle \cdot, \cdot \rangle)$ is a finite-dimensional inner product space over $\mathbb{F}$, then $\mathscr{V}$ has an orthonormal basis.*

Corollary 1.15. *Let $(\mathscr{V}, \langle \cdot, \cdot \rangle)$ be a finite-dimensional inner product space over $\mathbb{F}$ and $T \in \mathscr{L}(\mathscr{V})$. Then there exists an orthonormal basis $\mathscr{B}$ such that $M_{\mathscr{B}}^{\mathscr{B}}(T)$ is upper-triangular.*

Definition 1.16. Let $(\mathscr{V}, \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{F}$ and let $\mathscr{A} \subset \mathscr{V}$. We define the orthogonal complement of $\mathscr{A}$ to be

$$\mathscr{A}^{\perp} = \bigl\{ v \in \mathscr{V} : \langle v, a \rangle = 0 \; \forall \, a \in \mathscr{A} \bigr\}.$$

◊

The following is a straightforward proposition.

Proposition 1.17. *Let $(\mathscr{V}, \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{F}$ and $\mathscr{A} \subset \mathscr{V}$. Then $\mathscr{A}^{\perp}$ is a subspace of $\mathscr{V}$.*

Theorem 1.18. *Let $(\mathscr{V}, \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{F}$ and let $\mathscr{U}$ be a finite-dimensional subspace of $\mathscr{V}$. Then $\mathscr{V} = \mathscr{U} \oplus \mathscr{U}^{\perp}$.*

Proof. We first prove that $\mathscr{V} = \mathscr{U} + \mathscr{U}^{\perp}$. Note that since $\mathscr{U}$ is a subspace of $\mathscr{V}$, $\mathscr{U}$ inherits the inner product from $\mathscr{V}$. Thus $\mathscr{U}$ is a finite-dimensional inner product space. Thus there exists an orthonormal basis of $\mathscr{U}$, $\mathscr{B} = \{u_1, u_2, \dots u_k\}$.

Let $v \in \mathscr{V}$ be arbitrary. Then

$$v = \left(\sum_{j=1}^{k} \langle v, u_j \rangle u_j \right) + \left(v - \sum_{j=1}^{k} \langle v, u_j \rangle u_j \right),$$

where the first summand is in $\mathscr{U}$. By Theorem 1.11(b) the second summand is in $\mathscr{U}^{\perp}$. This proves that $\mathscr{V} = \mathscr{U} + \mathscr{U}^{\perp}$.

To prove that the sum is direct, let $w \in \mathscr{U}$ and $w \in \mathscr{U}^{\perp}$. Then $\langle w, w \rangle = 0$. Since $\langle \cdot, \cdot \rangle$ satisfies property IPD in Definition 1.1, $\langle w, w \rangle = 0$ implies $w = 0_{\mathscr{V}}$. The theorem is proved. □

Corollary 1.19. *Let $(\mathscr{V}, \langle\cdot\,,\cdot\rangle)$ be an inner product space over $\mathbb{F}$ and let $\mathscr{U}$ be a finite-dimensional subspace of $\mathscr{V}$. Then $\left(\mathscr{U}^{\perp}\right)^{\perp} = \mathscr{U}$.*

Recall that an arbitrary direct sum $\mathscr{V} = \mathscr{U} \oplus \mathscr{W}$ gives rise to a projection operator $P_{\mathscr{U}\|\mathscr{W}}$, the projection of $\mathscr{V}$ onto $\mathscr{U}$ parallel to $\mathscr{W}$.

If $\mathscr{V} = \mathscr{U} \oplus \mathscr{U}^{\perp}$, then the resulting projection of $\mathscr{V}$ onto $\mathscr{U}$ parallel to $\mathscr{U}^{\perp}$ is called the **orthogonal projection** of $\mathscr{V}$ onto $\mathscr{U}$; it is denoted simply by $P_{\mathscr{U}}$. By definition for every $v \in \mathscr{V}$,

$$u = P_{\mathscr{U}} v \qquad \Leftrightarrow \qquad u \in \mathscr{U} \quad \text{and} \quad v - u \in \mathscr{U}^{\perp}.$$

As for any projection we have $P_{\mathscr{U}} \in \mathscr{L}(\mathscr{V})$, $\operatorname{ran} P_{\mathscr{U}} = \mathscr{U}$, $\operatorname{nul} P_{\mathscr{U}} = \mathscr{U}^{\perp}$, and $(P_{\mathscr{U}})^2 = P_{\mathscr{U}}$.

Theorem 1.18 yields the following solution of the best approximation problem for finite-dimensional subspaces of a positive definite inner product space.

Corollary 1.20. *Let $(\mathscr{V}, \langle\cdot\,,\cdot\rangle)$ be an inner product space over $\mathbb{F}$ and let $\mathscr{U}$ be a finite-dimensional subspace of $\mathscr{V}$. For arbitrary $v \in \mathscr{V}$ the vector $P_{\mathscr{U}} v \in \mathscr{U}$ is the unique best approximation for v in $\mathscr{U}$. That is*

$$\left\| v - P_{\mathscr{U}} v \right\| < \| v - u \| \quad \text{for all} \quad u \in \mathscr{U} \setminus \left\{ P_{\mathscr{U}} v \right\}. \tag{1.9}$$

Proof. Let $v \in \mathscr{V}$ and $u \in \mathscr{U} \setminus \left\{ P_{\mathscr{U}} v \right\}$ be arbitrary. Recall the basic two facts which characterize the orthogonal projection $P_{\mathscr{U}} v$:

$$P_{\mathscr{U}} v \in \mathscr{U} \quad \text{and} \quad v - P_{\mathscr{U}} v \in \mathscr{U}^{\perp}.$$

In the next calculation we use the preceding two facts, the Pythagorean Theorem and the fact that $u \neq P_{\mathscr{U}} v$ as follows

$$\begin{aligned} \|v - u\|^2 &= \left\| v - P_{\mathscr{U}} v + P_{\mathscr{U}} v - u \right\|^2 \\ &= \left\| v - P_{\mathscr{U}} v \right\|^2 + \left\| P_{\mathscr{U}} v - u \right\|^2 \\ &> \left\| v - P_{\mathscr{U}} v \right\|^2. \end{aligned}$$

Taking the square root of both sides of the preceding inequality proves (1.9) in the corollary. □

2. The definition of an adjoint operator

Let $\mathscr{V}$ be a vector space over $\mathbb{F}$. The space $\mathscr{L}(\mathscr{V}, \mathbb{F})$ is called the **dual space** of $\mathscr{V}$; it is denoted by $\mathscr{V}^*$.

Definition 2.1. Let $\mathscr{V}$ and $\mathscr{W}$ be vector spaces over $\mathbb{F}$. A function $\Psi : \mathscr{V} \to \mathscr{W}$ is said to be **conjugate-linear** if for all $\alpha, \beta \in \mathbb{F}$ and all $u, v \in \mathscr{V}$ we have

$$\Psi(\alpha u + \beta v) = \overline{\alpha}\Psi(u) + \overline{\beta}\Psi(v).$$

◊

Theorem 2.2. *Let $\mathscr{V}$ be a finite-dimensional vector space over $\mathbb{F}$ with a positive definite inner product $\langle\,\cdot\,,\cdot\,\rangle$. Define the function*

$$\Phi : \mathscr{V} \to \mathscr{V}^*$$

as follows: for $w \in \mathscr{V}$ we set

$$\big(\Phi(w)\big)(v) = \langle v, w\rangle \quad \text{for all} \quad v \in \mathscr{V}.$$

Then Φ is an conjugate-linear bijection.

Proof. Clearly, for each $w \in \mathscr{V}$, $\Phi(w) \in \mathscr{V}^*$. The mapping Φ is conjugate-linear, since for $\alpha, \beta \in \mathbb{F}$ and $u, w \in \mathscr{V}$, for all $v \in \mathscr{V}$ we have

$$\begin{aligned}
\big(\Phi(\alpha u + \beta w)\big)(v) &= \langle v, \alpha u + \beta w\rangle \\
&= \overline{\alpha}\langle v, u\rangle + \overline{\beta}\langle v, w\rangle \\
&= \overline{\alpha}\big(\Phi(u)\big)(v) + \overline{\beta}\big(\Phi(w)\big)(v) \\
&= \big(\overline{\alpha}\Phi(u) + \overline{\beta}\Phi(w)\big)(v).
\end{aligned}$$

Thus $\Phi(\alpha u + \beta w) = \overline{\alpha}\Phi(u) + \overline{\beta}\Phi(w)$. This proves conjugate-linearity.

To prove injectivity of Φ, let $u, w \in \mathscr{V}$ be such that $\Phi(u) = \Phi(w)$. Then $\big(\Phi(u)\big)(v) = \big(\Phi(w)\big)(v)$ for all $v \in \mathscr{V}$. By the definition of Φ this means $\langle v, u\rangle = \langle v, w\rangle$ for all $v \in \mathscr{V}$. Consequently, $\langle v, u - w\rangle = 0$ for all $v \in \mathscr{V}$. In particular, with $v = u - w$ we have $\langle u - w, u - w\rangle = 0$. Since $\langle\,\cdot\,,\cdot\,\rangle$ is a positive definite inner product, it follows that $u - w = 0_{\mathscr{V}}$, that is $u = w$.

To prove that Φ is a surjection we use the assumption that $\mathscr{V}$ is finite-dimensional. Then there exists an orthonormal basis $(u_1, \ldots, u_n)$ of $\mathscr{V}$. Let $\varphi \in \mathscr{V}^*$ be arbitrary. Set

$$w = \sum_{j=1}^{n} \overline{\varphi(u_j)}u_j.$$

The proof that $\Phi(w) = \varphi$ follows. Let $v \in \mathscr{V}$ be arbitrary.

$$\begin{aligned}
\big(\Phi(w)\big)(v) &= \langle v, w\rangle \\
&= \left\langle v, \sum_{j=1}^{n} \overline{\varphi(u_j)}u_j \right\rangle \\
&= \sum_{j=1}^{n} \varphi(u_j)\langle v, u_j\rangle \\
&= \sum_{j=1}^{n} \langle v, u_j\rangle\varphi(u_j) \\
&= \varphi\left(\sum_{j=1}^{n} \langle v, u_j\rangle u_j\right) \\
&= \varphi(v).
\end{aligned}$$

Since the equality $\bigl(\Phi(w)\bigr)(v) = \varphi(v)$ holds for all $v \in \mathscr{V}$, we have proved $\Phi(w) = \varphi$. The theorem is proved. $\square$

Proposition 2.3. *Let $\mathscr{V}$ and $\mathscr{W}$ be vector spaces over $\mathbb{F}$ and let $\mathscr{V}$ be finite-dimensional. If $\Psi : \mathscr{V} \to \mathscr{W}$ is a conjugate-linear bijection, then $\mathscr{W}$ is finite-dimensional and $\dim \mathscr{V} = \dim \mathscr{W}$.*

Proof. Let $n = \dim \mathscr{V}$ and let $(u_1, \dots, u_n)$ be a basis for $\mathscr{V}$. We will prove that $\bigl(\Psi(u_1), \dots, \Psi(u_n)\bigr)$ is a basis for $\mathscr{W}$. First we prove that $\Psi(u_1), \dots, \Psi(u_n)$ are linearly independent. For this goal, let $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ be such that

$$\alpha_1 \Psi(u_1) + \cdots + \alpha_n \Psi(u_n) = 0_{\mathscr{W}}.$$

Since $\Psi : \mathscr{V} \to \mathscr{W}$ is conjugate-linear, the last equality is equivalent to

$$\Psi\bigl(\overline{\alpha_1} u_1 + \cdots + \overline{\alpha_n} u_n\bigr) = 0_{\mathscr{W}}.$$

Since conjugate-linearity of Ψ implies $\Psi(0_{\mathscr{V}}) = 0_{\mathscr{W}}$, and since Ψ is a bijection, we deduce

$$\overline{\alpha_1} u_1 + \cdots + \overline{\alpha_n} u_n = 0_{\mathscr{V}}.$$

Since $u_1, \dots, u_n$ are linearly independent, the last equality implies that for all $k \in \{1, \dots, n\}$ we have $\overline{\alpha_k} = 0_{\mathbb{F}}$. Therefore for all $k \in \{1, \dots, n\}$ we have $\alpha_k = \overline{\overline{\alpha_k}} = \overline{0_{\mathbb{F}}} = 0_{\mathbb{F}}$. This proves linear independence.

Now we prove that $\Psi(u_1), \dots, \Psi(u_n)$ span $\mathscr{W}$. Let $w \in \mathscr{W}$ be arbitrary. Since $\Psi : \mathscr{V} \to \mathscr{W}$ is a surjection there exists $v \in \mathscr{V}$ such that $\Psi(v) = w$. Since the vectors $u_1, \dots, u_n$ span $\mathscr{V}$, there exist $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ such that

$$v = \alpha_1 u_1 + \cdot + \alpha_n u_n.$$

Applying Ψ to both sides of the preceding equality and using that Ψ is conjugate-linear, we obtain

$$w = \Psi(v) = \Psi\bigl(\alpha_1 u_1 + \cdot + \alpha_n u_n\bigr) = \overline{\alpha_1} \Psi(u_1) + \cdots + \overline{\alpha_n} \Psi(u_n).$$

Thus, w is a linear combination of $\Psi(u_1), \dots, \Psi(u_n)$. Since $w \in \mathscr{W}$ was arbitrary, the vectors $\Psi(u_1), \dots, \Psi(u_n)$ span $\mathscr{W}$. This proves that the vectors $\Psi(u_1), \dots, \Psi(u_n)$ form a basis for $\mathscr{W}$. Thus $\dim \mathscr{V} = \dim \mathscr{W}$. $\square$

Corollary 2.4. *Let $(\mathscr{V}, \langle \cdot, \cdot \rangle)$ be a finite-dimensional inner product space over $\mathbb{F}$. Then $\dim \mathscr{V} = \dim \mathscr{V}^*$.*

Proof. Since $\Phi : \mathscr{V} \to \mathscr{V}^*$ from Theorem 2.2 is an conjugate-linear bijection, Proposition 2.3 implies that $\dim \mathscr{V} = \dim \mathscr{V}^*$. $\square$

The function Φ from Theorem 2.2 is a convenient tool for defining the adjoint of a linear operator. In the following definition, we will deal with two inner product spaces $(\mathscr{V}, \langle \cdot, \cdot \rangle_{\mathscr{V}})$ and $(\mathscr{W}, \langle \cdot, \cdot \rangle_{\mathscr{W}})$. We will use subscripts to emphasize different inner products and different functions Φ:

$$\Phi_{\mathscr{V}} : \mathscr{V} \to \mathscr{V}^*, \qquad \Phi_{\mathscr{W}} : \mathscr{W} \to \mathscr{W}^*.$$

Recall that for every $x, v \in \mathscr{V}$ we have

$$\bigl(\Phi_{\mathscr{V}}(v)\bigr)(x) = \langle x, v \rangle_{\mathscr{V}},$$

and for every $y, w \in \mathscr{W}$ we have

$$\big(\Phi_{\mathscr{W}}(w)\big)(y) = \langle y, w\rangle_{\mathscr{W}}.$$

Definition 2.5. Let $\big(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{V}}\big)$ and $\big(\mathscr{W}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{W}}\big)$ be two finite-dimensional inner product spaces over $\mathbb{F}$. Let $T \in \mathscr{L}(\mathscr{V}, \mathscr{W})$. We define the adjoint $T^* : \mathscr{W} \to \mathscr{V}$ of T by

$$T^* w \stackrel{\text{def}}{=} \Phi_{\mathscr{V}}^{-1}\big(\Phi_{\mathscr{W}}(w) \circ T\big), \qquad w \in \mathscr{W}. \tag{2.1}$$

$\Diamond$

Theorem 2.6. *Let $\big(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{V}}\big)$ and $\big(\mathscr{W}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{W}}\big)$ be two finite-dimensional inner product spaces over $\mathbb{F}$. If $T \in \mathscr{L}(\mathscr{V}, \mathscr{W})$, then $T^* \in \mathscr{L}(\mathscr{W}, \mathscr{V})$.*

Proof. Since $\Phi_{\mathscr{W}}$ and $\Phi_{\mathscr{V}}^{-1}$ are conjugate-linear, it follows that T^* is linear. Indeed, for arbitrary $\alpha_1, \alpha_2 \in \mathbb{F}$ and $w_1, w_2 \in \mathscr{W}$ we have

$$\begin{aligned}
T^*(\alpha_1 w_1 + \alpha_2 w_2) &= \Phi_{\mathscr{V}}^{-1}\big(\Phi_{\mathscr{W}}(\alpha_1 w_1 + \alpha_2 w_2) \circ T\big) \\
&= \Phi_{\mathscr{V}}^{-1}\big(\big(\overline{\alpha}_1 \Phi_{\mathscr{W}}(w_1) + \overline{\alpha}_2 \Phi_{\mathscr{W}}(w_2)\big) \circ T\big) \\
&= \Phi_{\mathscr{V}}^{-1}\big(\overline{\alpha}_1\, \Phi_{\mathscr{W}}(w_1) \circ T + \overline{\alpha}_2\, \Phi_{\mathscr{W}}(w_2) \circ T\big) \\
&= \alpha_1\, \Phi_{\mathscr{V}}^{-1}\big(\Phi_{\mathscr{W}}(w_1) \circ T\big) + \alpha_2\, \Phi_{\mathscr{W}}^{-1}\big(\Phi_{\mathscr{W}}(w_2) \circ T\big) \\
&= \alpha_1\, T^* w_1 + \alpha_2\, T^* w_2.
\end{aligned}$$

Thus, $T^* \in \mathscr{L}(\mathscr{W}, \mathscr{V})$. $\square$

Next we will deduce the most important characterization of T^*.

Theorem 2.7. *Let $\big(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{V}}\big)$ and $\big(\mathscr{W}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{W}}\big)$ be finite-dimensional inner product spaces over $\mathbb{F}$. Let $T \in \mathscr{L}(\mathscr{V}, \mathscr{W})$ and $S \in \mathscr{L}(\mathscr{W}, \mathscr{V})$. Then $S = T^*$ if and only if*

$$\langle Tv, w\rangle_{\mathscr{W}} = \langle v, Sw\rangle_{\mathscr{V}} \quad \textit{for all} \quad v \in \mathscr{V}, w \in \mathscr{W}. \tag{2.2}$$

Proof. First, we assume $S = T^*$ and prove (2.2). By the definition of $T^* : \mathscr{W} \to \mathscr{V}$, for a an arbitrary $w \in \mathscr{W}$ we have

$$T^* w = \Phi_{\mathscr{V}}^{-1}\big(\Phi_{\mathscr{W}}(w) \circ T\big).$$

Consequently,

$$\Phi_{\mathscr{V}}\big(T^* w\big) = \Phi_{\mathscr{W}}(w) \circ T.$$

By the definition of $\Phi_{\mathscr{V}}$, the last equality yields

$$\big(\Phi_{\mathscr{W}}(w) \circ T\big)(v) = \langle v, T^* w\rangle_{\mathscr{V}} \quad \text{for all} \quad v \in \mathscr{V}.$$

By the definition of the composition of functions

$$\big(\Phi_{\mathscr{W}}(w)\big)(Tv) = \langle v, T^* w\rangle_{\mathscr{V}} \quad \text{for all} \quad v \in \mathscr{V}.$$

By the definition of $\Phi_{\mathscr{W}}$, the last statement is equivalent to

$$\langle Tv, w\rangle_{\mathscr{W}} = \langle v, T^* w\rangle_{\mathscr{V}} \quad \text{for all} \quad v \in \mathscr{V}.$$

Thus, T^* satisfies (2.2).

To prove the converse, assume (2.2). Then, by the definition of $\Phi_{\mathscr{W}}$, for all $v \in \mathscr{V}$ and all $w \in \mathscr{W}$ we have $\bigl(\Phi_{\mathscr{W}}(w)\bigr)(Tv) = \langle v, Sw\rangle_{\mathscr{V}}$, which by the definitions of the composition of functions and $\Phi_{\mathscr{V}}$ yields $\bigl(\Phi_{\mathscr{W}}(w) \circ T\bigr)(v) = \Phi_{\mathscr{V}}\bigl(Sw\bigr)(v)$. Hence $\Phi_{\mathscr{V}}\bigl(Sw\bigr) = \Phi_{\mathscr{W}}(w) \circ T$ for all $w \in \mathscr{W}$. Since $\Phi_{\mathscr{V}}$ is invertible, applying $\Phi_{\mathscr{V}}^{-1}$ to both sides of the last equality yields $Sw = \Phi_{\mathscr{V}}^{-1}\bigl(\Phi_{\mathscr{W}}(w) \circ T\bigr) = T^*w$ for all $w \in \mathscr{W}$. Thus, $S = T^*$. □

3. Properties of the adjoint operator

Theorem 3.1. *Let $\bigl(\mathscr{U}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{U}}\bigr)$, $\bigl(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{V}}\bigr)$ and $\bigl(\mathscr{W}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{W}}\bigr)$ be finite-dimensional inner product spaces $\mathbb{F}$. Let $S \in \mathscr{L}(\mathscr{U},\mathscr{V})$ and $T \in \mathscr{L}(\mathscr{V},\mathscr{W})$. Then $(TS)^* = S^*T^*$.*

Proof. By definition for every $u \in \mathscr{U}$, $v \in \mathscr{V}$ and $w \in \mathscr{W}$ we have

$$
\begin{aligned}
S^*v &= \Phi_{\mathscr{U}}^{-1}\bigl(\Phi_{\mathscr{V}}(v) \circ S\bigr)\\
T^*w &= \Phi_{\mathscr{V}}^{-1}\bigl(\Phi_{\mathscr{W}}(w) \circ T\bigr)\\
(TS)^*w &= \Phi_{\mathscr{U}}^{-1}\bigl(\Phi_{\mathscr{W}}(w) \circ (TS)\bigr)
\end{aligned}
$$

With this, for arbitrary $w \in \mathscr{W}$ we calculate

$$
\begin{aligned}
S^*T^*w &= S^*(T^*w)\\
&= \Phi_{\mathscr{U}}^{-1}\Bigl(\Phi_{\mathscr{V}}\bigl(\Phi_{\mathscr{V}}^{-1}\bigl(\Phi_{\mathscr{W}}(w) \circ T\bigr)\bigr) \circ S\Bigr)\\
&= \Phi_{\mathscr{U}}^{-1}\Bigl(\Phi_{\mathscr{W}}(w) \circ T \circ S\Bigr)\\
&= \Phi_{\mathscr{U}}^{-1}\Bigl(\Phi_{\mathscr{W}}(w) \circ TS\Bigr)\\
&= (TS)^*w.
\end{aligned}
$$

Thus $(TS)^* = S^*T^*$. □

A function $f : X \to X$ is said to be an *involution* if it is its own inverse, that is, if $f(f(x)) = x$ for all $x \in X$.

Theorem 3.2. *Let $\bigl(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{V}}\bigr)$ and $\bigl(\mathscr{W}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{W}}\bigr)$ be finite-dimensional inner product spaces over $\mathbb{F}$. Then the adjoint mapping*

$$
{}^* : \mathscr{L}(\mathscr{V},\mathscr{W}) \to \mathscr{L}(\mathscr{W},\mathscr{V})
$$

is a conjugate-linear bijection. Its inverse is the adjoint mapping from $\mathscr{L}(\mathscr{W},\mathscr{V})$ to $\mathscr{L}(\mathscr{V},\mathscr{W})$. In particular, the adjoint mapping in $\mathscr{L}(\mathscr{V},\mathscr{V})$ is a conjugate-linear involution.

Proof. To prove that ${}^* : \mathscr{L}(\mathscr{V},\mathscr{W}) \to \mathscr{L}(\mathscr{W},\mathscr{V})$ is conjugate-linear let $\alpha, \beta \in \mathbb{F}$ be arbitrary and let $S, T \in \mathscr{L}(\mathscr{V},\mathscr{W})$ be arbitrary. By the definition of ${}^* : \mathscr{L}(\mathscr{V},\mathscr{W}) \to \mathscr{L}(\mathscr{W},\mathscr{V})$, for an arbitrary $w \in \mathscr{W}$ we have

$$
\begin{aligned}
(\alpha S + \beta T)^*w &= \Phi_{\mathscr{V}}^{-1}\bigl(\Phi_{\mathscr{W}}(w) \circ (\alpha S + \beta T)\bigr)\\
&= \Phi_{\mathscr{V}}^{-1}\bigl(\alpha\Phi_{\mathscr{W}}(w) \circ S + \beta\Phi_{\mathscr{W}}(w) \circ T\bigr)
\end{aligned}
$$

$$\begin{aligned} &= \overline{\alpha}\Phi_{\mathscr{V}}^{-1}\big(\Phi_{\mathscr{W}}(w)\circ S\big) + \overline{\beta}\Phi_{\mathscr{V}}^{-1}\big(\Phi_{\mathscr{W}}(w)\circ T\big) \\ &= \overline{\alpha}S^*w + \overline{\beta}T^*w \\ &= \big(\overline{\alpha}S^* + \overline{\beta}T^*\big)w. \end{aligned}$$

Hence, $(\alpha S + \beta T)^* = \overline{\alpha}S^* + \overline{\beta}T^*$.

To prove that the adjoint mapping $\,^* : \mathscr{L}(\mathscr{V},\mathscr{W}) \to \mathscr{L}(\mathscr{W},\mathscr{V})$ is a bijection we will use the adjoint mapping $\,^\star : \mathscr{L}(\mathscr{W},\mathscr{V}) \to \mathscr{L}(\mathscr{V},\mathscr{W})$. In fact we will prove that $\,^\star$ is the inverse of $\,^*$. To this end we will prove that for all $S \in \mathscr{L}(\mathscr{V},\mathscr{W})$ we have that $(S^*)^\star = S$ and that for all $T \in \mathscr{L}(\mathscr{W},\mathscr{V})$ we have that $(T^\star)^* = T$.

Here are the proofs. By the definition of the mapping $\,^* : \mathscr{L}(\mathscr{V},\mathscr{W}) \to \mathscr{L}(\mathscr{W},\mathscr{V})$ for an arbitrary $S \in \mathscr{L}(\mathscr{V},\mathscr{W})$ we have

$$\forall v \in \mathscr{V} \ \ \forall w \in \mathscr{W} \quad \langle S^*w, v\rangle_{\mathscr{V}} = \langle w, Sv\rangle_{\mathscr{W}}.$$

By Theorem 2.7 this identity yields $(S^*)^\star = S$. By the definition of the mapping $\,^\star : \mathscr{L}(\mathscr{W},\mathscr{V}) \to \mathscr{L}(\mathscr{V},\mathscr{W})$ for an arbitrary $T \in \mathscr{L}(\mathscr{W},\mathscr{V})$ we have

$$\forall w \in \mathscr{W} \ \ \forall v \in \mathscr{V} \quad \langle T^\star v, w\rangle_{\mathscr{W}} = \langle v, Tw\rangle_{\mathscr{V}}.$$

By Theorem 2.7, this identity yields $(T^\star)^* = T$. $\square$

Theorem 3.3. *Let $\big(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{V}}\big)$ and $\big(\mathscr{W}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{W}}\big)$ be a finite-dimensional inner product spaces over $\mathbb{F}$ and let $T \in \mathscr{L}(\mathscr{V},\mathscr{W})$. The following statements hold.*

(i) $\operatorname{nul}(T^*) = (\operatorname{ran} T)^\perp$.
(ii) $\operatorname{ran}(T^*) = (\operatorname{nul} T)^\perp$.
(iii) $\operatorname{nul}(T) = (\operatorname{ran} T^*)^\perp$.
(iv) $\operatorname{ran}(T) = (\operatorname{nul} T^*)^\perp$.

Theorem 3.4. *Let $\big(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{V}}\big)$ and $\big(\mathscr{W}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{W}}\big)$ be finite-dimensional inner product spaces over $\mathbb{F}$. Let $\mathscr{B}$ and $\mathscr{C}$ be orthonormal bases of $\big(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{V}}\big)$ and $\big(\mathscr{W}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{W}}\big)$, respectively, and let $T \in \mathscr{L}(\mathscr{V},\mathscr{W})$. Then the matrix $M_{\mathscr{B}}^{\mathscr{C}}(T^*)$ is the conjugate transpose of the matrix $M_{\mathscr{C}}^{\mathscr{B}}(T)$.*

Proof. Let $\mathscr{B} = \{v_1, \ldots, v_m\}$ and $\mathscr{C} = \{w_1, \ldots, w_n\}$ be orthonormal bases from the theorem. Let $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$. Then the term in the j-th column and the i-th row of the $n \times m$ matrix $M_{\mathscr{C}}^{\mathscr{B}}(T)$ is $\langle Tv_j, w_i\rangle$, while the term in the i-th column and the j-th row of the $m \times n$ matrix $M_{\mathscr{B}}^{\mathscr{C}}(T^*)$ is

$$\langle T^*w_i, v_j\rangle_{\mathscr{V}} = \langle w_i, Tv_j\rangle_{\mathscr{W}} = \overline{\langle Tv_j, w_i\rangle_{\mathscr{W}}}.$$

This proves the claim. $\square$

Proposition 3.5. *Let $\mathscr{V}$ be a vector space over $\mathbb{F}$ and let $\langle\,\cdot\,,\cdot\,\rangle_{\mathscr{V}}$ be a positive definite inner product on $\mathscr{V}$. Let $\mathscr{U}$ be a subspace of $\mathscr{V}$ and let $T \in \mathscr{L}(\mathscr{V})$. The subspace $\mathscr{U}$ is invariant under T if and only if the subspace $\mathscr{U}^\perp$ is invariant under T^*.*

Proof. By the definition of adjoint we have

$$\langle Tu, v\rangle_{\mathscr{V}} = \langle u, T^*v\rangle_{\mathscr{V}} \tag{3.1}$$

for all $u, v \in \mathscr{V}$. Assume $T\mathscr{U} \subseteq \mathscr{U}$. From (3.1) we get

$$0 = \langle Tu, v\rangle_{\mathscr{V}} = \langle u, T^*v\rangle_{\mathscr{V}} \qquad \forall\, u \in \mathscr{U} \quad \text{and} \quad \forall v \in \mathscr{U}^{\perp}.$$

Therefore, $T^*v \in \mathscr{U}^{\perp}$ for all $v \in \mathscr{U}^{\perp}$. This proves "only if" part.

The proof of the "if" part is similar. □

4. Intermezzo: Polarization Identity for Sesquilinear Forms

Definition 4.1. Let $\mathscr{V}$ be a vector space over $\mathbb{F}$. A function

$$[\,\cdot\,,\cdot\,] : \mathscr{V} \times \mathscr{V} \to \mathbb{F}$$

is a **sesquilinear form** on $\mathscr{V}$ if the following two conditions are satisfied.

(a) (linearity in the first variable)

$$\forall\, \alpha, \beta \in \mathbb{F} \quad \forall\, u, v, w \in \mathscr{V} \quad [\alpha u + \beta v, w] = \alpha[u, w] + \beta[v, w].$$

(b) (conjugate-linearity in the second variable)

$$\forall\, \alpha, \beta \in \mathbb{F} \quad \forall\, u, v, w \in \mathscr{V} \quad [u, \alpha v + \beta w] = \overline{\alpha}[u, v] + \overline{\beta}[u, w].$$

♢

Example 4.2. Let $\mathsf{M} \in \mathbb{C}^{n\times n}$ be arbitrary. Then

$$[\mathbf{x}, \mathbf{y}] = (\mathsf{M}\mathbf{x}) \cdot \mathbf{y}, \qquad \mathbf{x}, \mathbf{y} \in \mathbb{C}^n,$$

is a sesquilinear form on the complex vector space $\mathbb{C}^n$. Here $\cdot$ denotes the usual dot product in $\mathbb{C}$.

In general, if $\bigl(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle\bigr)$ is an inner product space and $T \in \mathscr{L}(\mathscr{V})$, then

$$[u, v] = \langle Tu, v\rangle, \quad u, v \in \mathscr{V},$$

is a sesquilinear form on $\mathscr{V}$. ♢

Theorem 4.3 (Polarization identity)**.** *Let $\mathscr{V}$ be a vector space over $\mathbb{C}$ and let $[\,\cdot\,,\cdot\,] : \mathscr{V} \times \mathscr{V} \to \mathbb{F}$ be a sesquilinear form on $\mathscr{V}$. Then*

$$[u, v] = \frac{1}{4}\sum_{k=0}^{3} \mathrm{i}^k \bigl[u + \mathrm{i}^k v, u + \mathrm{i}^k v\bigr] \tag{4.1}$$

for all $u, v \in \mathscr{V}$.

Proof. For the proof we expend the sum on the right hand side, ignoring the fraction $1/4$, using the linearity in the first variable and conjugate-linearity in the second variable. The resulting expression will have the following four values of the sesquilinear form: $[u, u]$, $[u, v]$, $[v, u]$, $[v, v]$. For each of these values and for each $k \in \{0, 1, 2, 3\}$ we present the corresponding coefficients in a table with the values of the form in the header and values for each k in each row:

	$[u,u]$	$[u,v]$	$[v,u]$	$[v,v]$
$k=0$	1	1	1	1
$k=1$	i	1	-1	i
$k=2$	-1	1	1	-1
$k=3$	$-$i	1	-1	$-$i
sum	0	4	0	0

□

Corollary 4.4. *Let $\mathscr{V}$ be a vector space over $\mathbb{C}$ and let $[\,\cdot\,,\cdot\,] : \mathscr{V} \times \mathscr{V} \to \mathbb{F}$ be a sesquilinear form on $\mathscr{V}$. If $[v,v] = 0$ for all $v \in \mathscr{V}$, then $[u,v] = 0$ for all $u, v \in \mathscr{V}$.*

5. Self-adjoint and normal operators

Definition 5.1. Let $(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle)$ be an inner product space over $\mathbb{F}$. An operator $T \in \mathscr{L}(\mathscr{V})$ is said to be **self-adjoint** if $T = T^*$. An operator $T \in \mathscr{L}(\mathscr{V})$ is said to be **normal** if $TT^* = T^*T$. ♢

Proposition 5.2. *Let $(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle)$ be a positive definite inner product space over $\mathbb{F}$. All eigenvalues of a self-adjoint $T \in \mathscr{L}(\mathscr{V})$ are real.*

Proof. Let $\lambda \in \mathbb{F}$ be an eigenvalue of T and let $Tv = \lambda v$ with a nonzero $v \in \mathscr{V}$. Then

$$\lambda\langle v,v\rangle = \langle Tv, v\rangle = \langle v, Tv\rangle = \langle v, \lambda v\rangle = \overline{\lambda}\langle v,v\rangle.$$

Since $\langle v,v\rangle > 0$ the preceding equalities yield $\lambda = \overline{\lambda}$. □

In the rest of this section we will consider only the scalar field $\mathbb{C}$.

Proposition 5.3 (this is 7.13 in the textbook)**.** *Let $(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle)$ be an inner product space over $\mathbb{C}$. Let $T \in \mathscr{L}(\mathscr{V})$. Then $T = 0_{\mathscr{L}(\mathscr{V})}$ if and only if $\langle Tv, v\rangle = 0$ for all $v \in \mathscr{V}$.*

Proof. Set, $[u,v] = \langle Tu, v\rangle$ for all $u, v \in \mathscr{V}$. Then $[\,\cdot\,,\cdot\,]$ is a sesquilinear form on $\mathscr{V}$. Since $\langle\,\cdot\,,\cdot\,\rangle$ is an inner product, $T = 0_{\mathscr{L}(\mathscr{V})}$ if and only if for all $u, v \in \mathscr{V}$ we have $\langle Tu, v\rangle = 0$, which in turn is equivalent to for all $u, v \in \mathscr{V}$ we have $[u,v] = 0$. By Corollary 4.4 $[u,v] = 0$ for all $u, v \in \mathscr{V}$ is equivalent to $[u,u] = 0$ for all $u \in \mathscr{V}$, that is to $\langle Tu, u\rangle = 0$ for all $u \in \mathscr{V}$. □

Proposition 5.4 (this is 7.14 in the textbook)**.** *Let $(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle)$ be a finite-dimensional inner product space over $\mathbb{C}$. An operator $T \in \mathscr{L}(\mathscr{V})$ is self-adjoint if and only if $\langle Tv, v\rangle \in \mathbb{R}$ for all $v \in \mathscr{V}$.*

Proof. An operator $T \in \mathscr{L}(\mathscr{V})$ is self-adjoint if and only if $T - T^* = 0_{\mathscr{V}}$. By Proposition 5.3, $T - T^* = 0_{\mathscr{V}}$ if and only if for all $v \in \mathscr{V}$ we have $\big\langle (T - T^*)v, v\big\rangle = 0$, which, in turn is equivalent to, for all $v \in \mathscr{V}$

$$\langle Tv, v\rangle = \langle T^*v, v\rangle = \langle v, Tv\rangle = \overline{\langle Tv, v\rangle}.$$

Since $\langle Tv, v\rangle = \overline{\langle Tv, v\rangle}$ if and only if $\langle Tv, v\rangle \in \mathbb{R}$, the proposition is proved. $\square$

Theorem 5.5 (this is 7.20 in the textbook)**.** *Let $(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle)$ be a finite-dimensional inner product space over $\mathbb{C}$. An operator $T \in \mathscr{L}(\mathscr{V})$ is normal if and only if $\|Tv\| = \|T^*v\|$ for all $v \in \mathscr{V}$.*

Proof. An operator $T \in \mathscr{L}(\mathscr{V})$ is normal if and only if $T^*T - TT^* = 0_{\mathscr{V}}$. By Proposition 5.3, $T^*T - TT^* = 0_{\mathscr{V}}$ if and only if for all $v \in \mathscr{V}$ we have $\big\langle (T^*T - TT^*)v, v\big\rangle = 0$, which, in turn is equivalent to, for all $v \in \mathscr{V}$

$$\langle Tv, Tv\rangle = \langle T^*Tv, v\rangle = \langle TT^*v, v\rangle = \langle T^*v, T^*v\rangle.$$

Since $\langle Tv, Tv\rangle = \|Tv\|^2$ and $\langle T^*v, T^*v\rangle = \|T^*v\|^2$, taking the square roots of the terms in the last displayed expression, proves the theorem. $\square$

Corollary 5.6 (this is 7.21 in the textbook)**.** *Let $(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle)$ be a finite-dimensional inner product space over $\mathbb{C}$ and let $T \in \mathscr{L}(\mathscr{V})$ be normal. Then for every $\lambda \in \mathbb{C}$ we have*

$$\operatorname{nul}\big(T^* - \overline{\lambda}I\big) = \operatorname{nul}(T - \lambda I).$$

In particular for all $\lambda \in \mathbb{C}$ we have that λ is an eigenvalue of T if and only if $\overline{\lambda}$ is an eigenvalue of T^*.

6. The Spectral Theorem

In the rest of the notes we will consider only the scalar field $\mathbb{C}$.

Theorem 6.1. *Let $(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle)$ be a finite-dimensional inner product space over $\mathbb{C}$. Let $T \in \mathscr{L}(\mathscr{V})$. Then $\mathscr{V}$ has an orthonormal basis which consists of eigenvectors of T if and only if T is normal. In other words, T is normal if and only if there exists an orthonormal basis $\mathscr{B}$ of $\mathscr{V}$ such that $M_{\mathscr{B}}^{\mathscr{B}}(T)$ is a diagonal matrix.*

Proof. Let $n = \dim(\mathscr{V})$. Assume that T is normal. By Corollary 1.15 there exists an orthonormal basis $\mathscr{B} = (u_1, \ldots, u_n)$ of $\mathscr{V}$ such that $M_{\mathscr{B}}^{\mathscr{B}}(T)$ is upper-triangular. That is,

$$M_{\mathscr{B}}^{\mathscr{B}}(T) = \begin{bmatrix} \langle Tu_1, u_1\rangle & \langle Tu_2, u_1\rangle & \cdots & \langle Tu_n, u_1\rangle \\ 0 & \langle Tu_2, u_2\rangle & \cdots & \langle Tu_n, u_2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \langle Tu_n, u_n\rangle \end{bmatrix}, \tag{6.1}$$

or, equivalently,

$$Tu_k = \sum_{j=1}^{k} \langle Tu_k, u_j\rangle u_j \quad \text{for all} \quad k \in \{1, \ldots, n\}. \tag{6.2}$$

By Theorem 3.4 we have

$$M_{\mathscr{B}}^{\mathscr{B}}(T^*) = \begin{bmatrix} \overline{\langle Tu_1, u_1\rangle} & 0 & \cdots & 0 \\ \overline{\langle Tu_2, u_1\rangle} & \overline{\langle Tu_2, u_2\rangle} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \overline{\langle Tu_n, u_1\rangle} & \overline{\langle Tu_n, u_2\rangle} & \cdots & \overline{\langle Tu_n, u_n\rangle} \end{bmatrix}.$$

Consequently,

$$T^* u_k = \sum_{j=k}^{n} \overline{\langle Tu_j, u_k\rangle} u_j \quad \text{for all} \quad k \in \{1, \dots, n\}. \tag{6.3}$$

Since T is normal, Theorem 5.5 implies

$$\|Tu_k\|^2 = \|T^* u_k\|^2 \quad \text{for all} \quad k \in \{1, \dots, n\}.$$

Together with (6.2) and (6.3) the last identities become

$$\sum_{j=1}^{k} \bigl|\langle Tu_k, u_j\rangle\bigr|^2 = \sum_{j=k}^{n} \bigl|\overline{\langle Tu_j, u_k\rangle}\bigr|^2 \quad \text{for all} \quad k \in \{1, \dots, n\},$$

or, equivalently,

$$\sum_{j=1}^{k} \bigl|\langle Tu_k, u_j\rangle\bigr|^2 = \sum_{j=k}^{n} \bigl|\langle Tu_j, u_k\rangle\bigr|^2 \quad \text{for all} \quad k \in \{1, \dots, n\}. \tag{6.4}$$

The equality in (6.4) corresponding to $k = 1$ reads

$$\bigl|\langle Tu_1, u_1\rangle\bigr|^2 = \bigl|\langle Tu_1, u_1\rangle\bigr|^2 + \sum_{j=2}^{n} \bigl|\langle Tu_j, u_1\rangle\bigr|^2,$$

which implies

$$\langle Tu_j, u_1\rangle = 0 \quad \text{for all} \quad j \in \{2, \dots, n\} \tag{6.5}$$

In other words we have proved that the off-diagonal entries in the first row of the upper triangular matrix $M_{\mathscr{B}}^{\mathscr{B}}(T)$ in (6.1) are all zero.

Substituting the value $\langle Tu_2, u_1\rangle = 0$ (from (6.5)) in the equality in (6.4) corresponding to $k = 2$ reads we get

$$\bigl|\langle Tu_2, u_2\rangle\bigr|^2 = \bigl|\langle Tu_2, u_2\rangle\bigr|^2 + \sum_{j=3}^{n} \bigl|\langle Tu_j, u_2\rangle\bigr|^2,$$

which implies

$$\langle Tu_j, u_2\rangle = 0 \quad \text{for all} \quad j \in \{3, \dots, n\} \tag{6.6}$$

In other words we have proved that the off-diagonal entries in the second row of the upper triangular matrix $M_{\mathscr{B}}^{\mathscr{B}}(T)$ in (6.1) are all zero.

Repeating this reasoning $n - 2$ more times would prove that all the off-diagonal entries of the upper triangular matrix $M_{\mathscr{B}}^{\mathscr{B}}(T)$ in (6.1) are zero. That is, $M_{\mathscr{B}}^{\mathscr{B}}(T)$ is a diagonal matrix.

To prove the converse, assume that there exists an orthonormal basis $\mathscr{B} = (u_1, \dots, u_n)$ of $\mathscr{V}$ which consists of eigenvectors of T. That is, for some $\lambda_j \in \mathbb{C}$,

$$Tu_j = \lambda_j u_j \quad \text{for all} \quad j \in \{1, \dots, n\},$$

Then, for arbitrary $v \in \mathscr{V}$ we have

$$Tv = T\left(\sum_{j=1}^{n} \langle v, u_j \rangle u_j\right) = \sum_{j=1}^{n} \langle v, u_j \rangle T u_j = \sum_{j=1}^{n} \lambda_j \langle v, u_j \rangle u_j. \tag{6.7}$$

Therefore, for arbitrary $k \in \{1, \dots, n\}$ we have

$$\langle Tv, u_k \rangle = \lambda_k \langle v, u_k \rangle. \tag{6.8}$$

Now we calculate

$$\begin{aligned} T^*Tv &= \sum_{j=1}^{n} \langle T^*Tv, u_j \rangle u_j \\ &= \sum_{j=1}^{n} \langle Tv, Tu_j \rangle u_j \\ &= \sum_{j=1}^{n} \langle Tv, Tu_j \rangle u_j \\ &= \sum_{j=1}^{n} \overline{\lambda}_j \langle Tv, u_j \rangle u_j \\ &= \sum_{j=1}^{n} \lambda_j \overline{\lambda}_j \langle v, u_j \rangle u_j. \end{aligned}$$

Similarly,

$$\begin{aligned} TT^*v &= T\left(\sum_{j=1}^{n} \langle T^*v, u_j \rangle u_j\right) \\ &= \sum_{j=1}^{n} \langle v, Tu_j \rangle Tu_j \\ &= \sum_{j=1}^{n} \langle v, \lambda_j u_j \rangle \lambda_j u_j \\ &= \sum_{j=1}^{n} \lambda_j \overline{\lambda}_j \langle v, u_j \rangle u_j. \end{aligned}$$

Thus, we proved $T^*Tv = TT^*v$, that is, T is normal. □

A different proof of the "only if" part of the spectral theorem for normal operators follows. In this proof we use δ_{ij} to represent the values of the

Kronecker delta function:

$$\delta : \mathbb{N} \times \mathbb{N} \to \{0, 1\}$$

such that for all $i, j \in \mathbb{N}$ we have $\delta_{ij} = 1$ if and only if $i = j$.

Another proof of Theorem 6.1. Set $n = \dim \mathscr{V}$. We prove the "only if" part. Assume that T is normal. Set

$$\mathbb{K} = \left\{ k \in \{1, \ldots, n\} \ : \ \begin{array}{l} \exists\, w_1, \ldots, w_k \in \mathscr{V} \quad \text{and} \quad \exists\, \lambda_1, \ldots, \lambda_k \in \mathbb{C} \\ \text{such that } \langle w_i, w_j \rangle = \delta_{ij} \text{ and } T w_j = \lambda_j w_j \\ \text{for all} \quad i, j \in \{1, \ldots, k\} \end{array} \right\}$$

Clearly $1 \in \mathbb{K}$. Since $\mathbb{K}$ is finite, $m = \max \mathbb{K}$ exists. Clearly, $m \leq n$.

Next we will prove that $k \in \mathbb{K}$ and $k < n$ implies that $k + 1 \in \mathbb{K}$. Assume $k \in \mathbb{K}$ and $k < n$. Let $w_1, \ldots, w_k \in \mathscr{V}$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ be such that $\langle w_i, w_j \rangle = \delta_{ij}$ and $T w_j = \lambda_j w_j$ for all $i, j \in \{1, \ldots, k\}$. Set

$$\mathscr{W} = \operatorname{span}\{w_1, \ldots, w_k\}.$$

Since $w_1, \ldots, w_k$ are eigenvectors of T we have $T\mathscr{W} \subseteq \mathscr{W}$. By Proposition 3.5, $T^*\big(\mathscr{W}^\perp\big) \subseteq \mathscr{W}^\perp$. Thus, $T^*|_{\mathscr{W}^\perp} \in \mathscr{L}\big(\mathscr{W}^\perp\big)$. Since $\dim \mathscr{W} = k < n$ we have $\dim\big(\mathscr{W}^\perp\big) = n - k \geq 1$. Since $\mathscr{W}^\perp$ is a complex vector space the operator $T^*|_{\mathscr{W}^\perp}$ has an eigenvalue μ with the corresponding unit eigenvector u. Clearly, $u \in \mathscr{W}^\perp$ and $T^* u = \mu u$. Since T^* is normal, Corollary 5.6 yields that $T u = \overline{\mu} u$. Since $u \in \mathscr{W}^\perp$ and $T u = \overline{\mu} u$, setting $w_{k+1} = u$ and $\lambda_{k+1} = \overline{\mu}$ we have

$$\langle w_i, w_j \rangle = \delta_{ij} \quad \text{and} \quad T w_j = \lambda_j w_j \quad \text{for all} \quad i, j \in \{1, \ldots, k, k+1\}.$$

Thus $k + 1 \in \mathbb{K}$. Consequently, $k < m$. Thus, for $k \in \mathbb{K}$, we have proved the implication

$$k < n \qquad \Rightarrow \qquad k < m.$$

The contrapositive of this implication is: For $k \in \mathbb{K}$, we have

$$k \geq m \qquad \Rightarrow \qquad k \geq n.$$

In particular, for $m \in \mathbb{K}$ we have $m = m$ implies $m \geq n$. Since $m \leq n$ is also true, this proves that $m = n$. That is, $n \in \mathbb{K}$. This implies that there exist $u_1, \ldots, u_n \in \mathscr{V}$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ such that $\langle u_i, u_j \rangle = \delta_{ij}$ and $T u_j = \lambda_j u_j$ for all $i, j \in \{1, \ldots, n\}$.

Since $u_1, \ldots, u_n$ are orthonormal, they are linearly independent. Since $n = \dim \mathscr{V}$, it turns out that $u_1, \ldots, u_n$ form a basis of $\mathscr{V}$. This completes the proof. $\square$

The proof of the next corollary uses the property of polynomials that for arbitrary distinct $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ and arbitrary $\beta_1, \ldots, \beta_m \in \mathbb{C}$ there exists a unique polynomial $p(z)$ of degree $m - 1$, that is $p(z) \in \mathbb{C}[z]_{<m}$, such that for all $j \in \{1, \ldots, m\}$ we have $p(\alpha_j) = \beta_j$.

Corollary 6.2. *Let $(\mathscr{V}, \langle \cdot , \cdot \rangle)$ be a finite-dimensional inner product space over $\mathbb{C}$ and let $T \in \mathscr{L}(\mathscr{V})$. Then T is normal if and only if there exists a polynomial $p(z) \in \mathbb{C}[z]$ such that $T^* = p(T)$.*

Proof. Assume T is normal. By Theorem 6.1 there exists an orthonormal basis $(u_1,\ldots,u_n)$ and $\{\lambda_1,\ldots,\lambda_n\}\subseteq\mathbb{C}$ such that

$$Tu_j=\lambda_j u_j \quad \text{for all} \quad j\in\{1,\ldots,n\}.$$

Consequently,

$$T^*u_j=\overline{\lambda}_j u_j \quad \text{for all} \quad j\in\{1,\ldots,n\}.$$

Let v be arbitrary in $\mathscr{V}$. Applying T and T^* to the expansion of v in the basis vectors $(u_1,\ldots,u_n)$ we obtain

$$Tv=\sum_{j=1}^{n}\lambda_j\langle v,u_j\rangle u_j$$

and

$$T^*v=\sum_{j=1}^{n}\overline{\lambda_j}\langle v,u_j\rangle u_j.$$

Let $\#\{\lambda_1,\ldots,\lambda_n\}=m$ where $m\in\mathbb{N}$. That is, assume that T has m distinct eigenvalues. Then there exists a unique polynomial $p(z)=a_0+a_1z+\cdots+a_{m-1}z^{m-1}\in\mathbb{C}[z]$ such that

$$p(\lambda_j)=\overline{\lambda}_j \quad \text{for all} \quad j\in\{1,\ldots,n\}.$$

Clearly, for all $j\in\{1,\ldots,n\}$ we have

$$p(T)u_j=p(\lambda_j)u_j=\overline{\lambda}_j u_j=T^*u_j.$$

Therefore $p(T)=T^*$.

The converse is straightforward. □

7. Invariance under a normal operator

Theorem 7.1. *Let $(\mathscr{V},\langle\,\cdot\,,\cdot\,\rangle)$ be a finite-dimensional inner product space over $\mathbb{C}$. Let $T\in\mathscr{L}(\mathscr{V})$ be normal and let $\mathscr{U}$ be a subspace of $\mathscr{V}$. Then*

$$T\mathscr{U}\subseteq\mathscr{U} \quad\Leftrightarrow\quad T\mathscr{U}^\perp\subseteq\mathscr{U}^\perp$$

Recall that we have previously proved that for any $T\in\mathscr{L}(\mathscr{V})$ we have

$$T\mathscr{U}\subseteq\mathscr{U}\Leftrightarrow T^*\mathscr{U}^\perp\subseteq\mathscr{U}^\perp.$$

So, the claim of the theorem gives an additional information about a normal operator.

We will give three proofs of this theorem. The **first proof** follows.

Proof of Theorem 7.1. Assume $T\mathscr{U}\subseteq\mathscr{U}$. We know $\mathscr{V}=\mathscr{U}\oplus\mathscr{U}^\perp$. Let $(u_1,\ldots,u_m)$ be an orthonormal basis of $\mathscr{U}$ and let $(u_{m+1},\ldots,u_n)$ be an orthonormal basis of $\mathscr{U}^\perp$. Then $(u_1,\ldots,u_n)$ is an orthonormal basis of $\mathscr{V}$. If $j\in\{1,\ldots,m\}$ then $u_j\in\mathscr{U}$, so $Tu_j\in\mathscr{U}$. Hence

$$Tu_j=\sum_{k=1}^{m}\langle Tu_j,u_k\rangle u_k.$$

Also, clearly,

$$T^* u_j = \sum_{k=1}^{n} \langle T^* u_j, u_k \rangle u_k.$$

Since T is normal, by Theorem 5.5 we have $\|Tu_j\|^2 = \|T^* u_j\|^2$ for all $j \in \{1, \dots, m\}$. Starting with this, we calculate

$$\begin{aligned}
\sum_{j=1}^{m} \|Tu_j\|^2 &= \sum_{j=1}^{m} \|T^* u_j\|^2 \\
\boxed{\text{Pythag. thm.}} &= \sum_{j=1}^{m} \sum_{k=1}^{n} |\langle T^* u_j, u_k \rangle|^2 \\
\boxed{\text{group terms}} &= \sum_{j=1}^{m} \sum_{k=1}^{m} |\langle T^* u_j, u_k \rangle|^2 + \sum_{j=1}^{m} \sum_{k=m+1}^{n} |\langle T^* u_j, u_k \rangle|^2 \\
\boxed{\text{def. of } T^*} &= \sum_{j=1}^{m} \sum_{k=1}^{m} |\langle u_j, Tu_k \rangle|^2 + \sum_{j=1}^{m} \sum_{k=m+1}^{n} |\langle T^* u_j, u_k \rangle|^2 \\
\boxed{|\alpha| = |\overline{\alpha}|} &= \sum_{j=1}^{m} \sum_{k=1}^{m} |\langle Tu_k, u_j \rangle|^2 + \sum_{j=1}^{m} \sum_{k=m+1}^{n} |\langle T^* u_j, u_k \rangle|^2 \\
\boxed{\text{order of sum.}} &= \sum_{k=1}^{m} \sum_{j=1}^{m} |\langle Tu_k, u_j \rangle|^2 + \sum_{j=1}^{m} \sum_{k=m+1}^{n} |\langle T^* u_j, u_k \rangle|^2 \\
\boxed{\text{Pythag. thm.}} &= \sum_{k=1}^{m} \|Tu_k\|^2 + \sum_{j=1}^{m} \sum_{k=m+1}^{n} |\langle T^* u_j, u_k \rangle|^2.
\end{aligned}$$

From the above equality we deduce that $\sum_{j=1}^{m} \sum_{k=m+1}^{n} |\langle T^* u_j, u_k \rangle|^2 = 0$. As each term is nonnegative, we conclude that $|\langle T^* u_j, u_k \rangle|^2 = |\langle u_j, Tu_k \rangle|^2 = 0$, that is,

$$\langle u_j, Tu_k \rangle = 0 \quad \text{for all} \quad j \in \{1, \dots, m\},\ k \in \{m+1, \dots, n\}. \tag{7.1}$$

Let now $w \in \mathscr{U}^{\perp}$ be arbitrary. Then

$$\begin{aligned}
Tw &= \sum_{j=1}^{n} \langle Tw, u_j \rangle u_j \\
\boxed{\text{use } w = \sum_{k=m+1}^{n} \langle w, u_k \rangle u_k} &= \sum_{j=1}^{n} \left\langle \sum_{k=m+1}^{n} \langle w, u_k \rangle Tu_k, u_j \right\rangle u_j \\
&= \sum_{j=1}^{n} \sum_{k=m+1}^{n} \langle w, u_k \rangle \langle Tu_k, u_j \rangle u_j \\
\boxed{\text{by (7.1)}} &= \sum_{j=m+1}^{n} \sum_{k=m+1}^{n} \langle w, u_k \rangle \langle Tu_k, u_j \rangle u_j
\end{aligned}$$

Hence $Tw \in \mathscr{U}^{\perp}$, that is $T\mathscr{U}^{\perp} \subseteq \mathscr{U}^{\perp}$. $\square$

The **second proof** follows.

Proof of Theorem 7.1. Assume T is normal. By Corollary 6.2 there exists a polynomial $p(z) = a_0 + a_1 z + \cdots + a_{m-1}z^{m-1} \in \mathbb{C}[z]$ $p(T) = T^*$. Assume that $T\mathscr{U} \subseteq \mathscr{U}$. Then $T^k\mathscr{U} \subseteq \mathscr{U}$ for all $k \in \mathbb{N}$ and also $\alpha T\mathscr{U} \subseteq \mathscr{U}$ for all $\alpha \in \mathbb{C}$. Hence $p(T)\mathscr{U} = T^*\mathscr{U} \subseteq \mathscr{U}$. Now, the theorem follows from Proposition 3.5. $\square$

Lastly, we review the proof in the book. This proof is in essence very similar to the first proof. It brings up a matrix representation of T, which might help us visualize what is going on in the first proof. The **third proof** follows.

Proof of Theorem 7.1. Assume $T\mathscr{U} \subseteq \mathscr{U}$. By Proposition 3.5, $T^*(\mathscr{U}^{\perp}) \subseteq \mathscr{U}^{\perp}$.

Now $\mathscr{V} = \mathscr{U} \oplus \mathscr{U}^{\perp}$. Let $n = \dim(\mathscr{V})$. Let $\{u_1, \ldots, u_m\}$ be an orthonormal basis of $\mathscr{U}$ and let $\{u_{m+1}, \ldots, u_n\}$ be an orthonormal basis of $\mathscr{U}^{\perp}$. Then $\mathscr{B} = \{u_1, \ldots, u_n\}$ is an orthonormal basis of $\mathscr{V}$. Since $Tu_j \in \mathscr{U}$ for all $j \in \{1, \ldots, m\}$ we have

$$M_{\mathscr{B}}^{\mathscr{B}}(T) = \begin{array}{cc}
\begin{array}{c} u_1 \\ \vdots \\ u_m \\ u_{m+1} \\ \vdots \\ u_n \end{array} & \left[\begin{array}{ccc|ccc} \langle Tu_1, u_1\rangle & \cdots & \langle Tu_m, u_1\rangle & & & \\ \vdots & \ddots & \vdots & & \mathsf{B} & \\ \langle Tu_1, u_m\rangle & \cdots & \langle Tu_m, u_m\rangle & & & \\ \hline & & & & & \\ & \mathsf{0} & & & \mathsf{C} & \\ & & & & & \end{array}\right] \\
 & \begin{array}{cccccc} Tu_1 & \cdots & Tu_m & Tu_{m+1} & \cdots & Tu_n \end{array}
\end{array} \tag{7.2}$$

Here we prepended the basis vectors on the left hand side of the matrix and we appended the images of the basis vectors under T below the matrix to emphasize that an appended vector Tu_k is expended as a linear combination of the basis vectors which are prepended with the coefficients given in the k-th column of the matrix.

For $k \in \{1, \ldots, m\}$ we have $Tu_k = \sum_{j=1}^{m} \langle Tu_k, u_j\rangle u_j$. By the Pythagorean Theorem

$$\|Tu_k\|^2 = \sum_{j=1}^{m} |\langle Tu_k, u_j\rangle|^2 \quad \text{and} \quad \|T^*u_k\|^2 = \sum_{j=1}^{n} |\langle T^*u_k, u_j\rangle|^2.$$

Since T is normal, $\|Tu_k\|^2 = \|T^*u_k\|^2$ for all $k \in \{1, \ldots, m\}$, and therefore $\sum_{k=1}^{m} \|Tu_k\|^2 = \sum_{k=1}^{m} \|T^*u_k\|^2$. Consequently,

$$\sum_{k=1}^{m}\sum_{j=1}^{m} |\langle Tu_k, u_j\rangle|^2 = \sum_{k=1}^{m}\sum_{j=1}^{n} |\langle T^*u_k, u_j\rangle|^2$$

$$= \sum_{k=1}^{m}\sum_{j=1}^{m} |\langle T^* u_k, u_j\rangle|^2 + \sum_{k=1}^{m}\sum_{j=m+1}^{n} |\langle T^* u_k, u_j\rangle|^2$$
$$= \sum_{k=1}^{m}\sum_{j=1}^{m} |\langle u_k, T u_j\rangle|^2 + \sum_{k=1}^{m}\sum_{j=m+1}^{n} |\langle T^* u_k, u_j\rangle|^2.$$

We have
$$\sum_{k=1}^{m}\sum_{j=1}^{m} |\langle T u_k, u_j\rangle|^2 = \sum_{k=1}^{m}\sum_{j=1}^{m} |\langle u_k, T u_j\rangle|^2$$
since these sums consist of identical terms. Hence, the last two displayed equalities yield
$$\sum_{k=1}^{m}\sum_{j=m+1}^{n} |\langle T^* u_k, u_j\rangle|^2 = 0$$
As the last double sum consists of the nonnegative terms we deduce that for all $k \in \{1,\dots,m\}$ and for all $j \in \{m+1,\dots,n\}$ we have
$$0 = |\langle T^* u_k, u_j\rangle|^2 = |\langle u_k, T u_j\rangle|^2 = |\langle T u_j, u_k\rangle|^2.$$
Hence also $\langle T u_j, u_k\rangle = 0$ for all $k \in \{1,\dots,m\}$ and for all $j \in \{m+1,\dots,n\}$. This proves that $B = 0$ in (7.2). Therefore, $T u_j$ is orthogonal to $\mathscr{U}$ for all $j \in \{m+1,\dots,n\}$, which implies $T\big(\mathscr{U}^{\perp}\big) \subseteq \mathscr{U}^{\perp}$. □

Theorem 7.1 and Proposition 3.5 yield the following corollary.

Corollary 7.2. *Let $(\mathscr{V}, \langle\cdot,\cdot\rangle)$ be a finite-dimensional inner product space over $\mathbb{C}$. Let $T \in \mathscr{L}(\mathscr{V})$ be normal and let $\mathscr{U}$ be a subspace of $\mathscr{V}$. The following statements are equivalent:*

(a) $T\mathscr{U} \subseteq \mathscr{U}$.
(b) $T(\mathscr{U}^{\perp}) \subseteq \mathscr{U}^{\perp}$.
(c) $T^*\mathscr{U} \subseteq \mathscr{U}$.
(d) $T^*(\mathscr{U}^{\perp}) \subseteq \mathscr{U}^{\perp}$.

If one (and hence all) of the above statements holds, then the following statements are true:

(e) $(T|_{\mathscr{U}})^* = T^*|_{\mathscr{U}}$.
(f) $(T|_{\mathscr{U}^{\perp}})^* = T^*|_{\mathscr{U}^{\perp}}$.
(g) $T|_{\mathscr{U}}$ *is normal on* $\mathscr{U}$.
(h) $T|_{\mathscr{U}^{\perp}}$ *is normal on* $\mathscr{U}^{\perp}$.

8. Polar Decomposition

There are two distinct subsets of $\mathbb{C}$. Those are the set of nonnegative real numbers, denoted by $\mathbb{R}_{\geq 0}$, and the set of complex numbers of modulus 1, denoted by $\mathbb{T}$. An important tool in complex analysis is the polar representation of a complex number: for every $\alpha \in \mathbb{C}$ there exists $r \in \mathbb{R}_{\geq 0}$ and $u \in \mathbb{T}$ such that $\alpha = r\,u$.

In this section we will prove that an analogous statement holds for operators in $\mathscr{L}(\mathscr{V})$, where $(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle)$ is an inner product space over $\mathbb{C}$. The initial step in proving this analogous result involves identifying operators in $\mathscr{L}(\mathscr{V})$ that correspond to nonnegative real numbers and identifying operators in $\mathscr{L}(\mathscr{V})$ that correspond to complex numbers with modulus 1. That is done in the following two definitions.

Definition 8.1. Let $(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle)$ be an inner product space over $\mathbb{C}$. An operator $Q \in \mathscr{L}(\mathscr{V})$ is said to be **nonnegative** if $\langle Qv, v\rangle \geq 0$ for all $v \in \mathscr{V}$. $\Diamond$

Note that Axler uses the term “positive” instead of nonnegative. We think that nonnegative is more appropriate, since $0_{\mathscr{L}(\mathscr{V})}$ is a nonnegative operator. There is nothing positive about any zero, we think.

Proposition 8.2. *Let $(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle)$ be a finite-dimensional inner product space over $\mathbb{C}$ and let $T \in \mathscr{L}(\mathscr{V})$. Then T is nonnegative if and only if T is normal and all its eigenvalues are nonnegative.*

Theorem 8.3. *Let $(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle)$ be a finite-dimensional inner product space over $\mathbb{C}$. Let $Q \in \mathscr{L}(\mathscr{V})$ be a nonnegative operator, let $(u_1, \dots, u_n)$ be an orthonormal basis of $\mathscr{V}$, and let $\lambda_1, \dots, \lambda_n \in \mathbb{R}_{\geq 0}$ be such that*

$$Qu_j = \lambda_j u_j \quad \text{for all} \quad j \in \{1, \dots, n\}. \tag{8.1}$$

The following statements are equivalent.

(a) *$S \in \mathscr{L}(\mathscr{V})$ is a nonnegative operator and $S^2 = Q$.*

(b) *For every $\lambda \in \mathbb{R}_{\geq 0}$ we have*

$$\operatorname{nul}(Q - \lambda I) = \operatorname{nul}(S - \sqrt{\lambda} I).$$

(c) *For every $v \in \mathscr{V}$ we have*

$$Sv = \sum_{j=1}^{n} \sqrt{\lambda_j} \langle v, u_j\rangle u_j.$$

Proof. (a) $\Rightarrow$ (b). We first prove that $\operatorname{nul} Q = \operatorname{nul} S$. Since $Q = S^2$ we have $\operatorname{nul} S \subseteq \operatorname{nul} Q$. Let $v \in \operatorname{nul} Q$, that is, let $Qv = S^2 v = 0$. Then $\langle S^2 v, v\rangle = 0$. Since S is nonnegative it is self-adjoint. Therefore, $\langle S^2 v, v\rangle = \langle Sv, Sv\rangle = \|Sv\|^2$. Hence, $\|Sv\| = 0$, and thus $Sv = 0$. This proves that $\operatorname{nul} Q \subseteq \operatorname{nul} S$ and (b) is proved for $\lambda = 0$.

Let $\lambda > 0$. Then the operator $S + \sqrt{\lambda} I$ is invertible. To prove this, let $v \in \mathscr{V} \setminus \{0_{\mathscr{W}}\}$ be arbitrary. Then $\|v\| > 0$ and therefore

$$\big\langle (S + \sqrt{\lambda} I)v, v\big\rangle = \langle Sv, v\rangle + \sqrt{\lambda}\langle v, v\rangle \geq \sqrt{\lambda}\|v\|^2 > 0.$$

Thus, $v \neq 0$ implies $(S + \sqrt{\lambda} I)v \neq 0$. This proves the injectivity of $S + \sqrt{\lambda} I$.

To prove $\operatorname{nul}(Q - \lambda I) = \operatorname{nul}(S - \sqrt{\lambda} I)$, let $v \in \mathscr{V}$ be arbitrary and notice that $(Q - \lambda I)v = 0$ if and only if $\big(S^2 - \sqrt{\lambda}^2 I\big)v = 0$, which, in turn, is equivalent to

$$\big(S + \sqrt{\lambda} I\big)\big(S - \sqrt{\lambda} I\big)v = 0.$$

Since $S+\sqrt{\lambda}I$ is injective, the last equality is equivalent to $\bigl(S-\sqrt{\lambda}I\bigr)v=0$. This completes the proof of (b).

(b) $\Rightarrow$ (c). Let $(u_1,\dots,u_n)$ be an orthonormal basis of $\mathscr{V}$ and let $\lambda_1,\dots,\lambda_n\in\mathbb{R}_{\geq 0}$ be such that (8.1) holds. For arbitrary $j\in\{1,\dots,n\}$ (8.1) yields $u_j\in\operatorname{nul}(Q-\lambda_j I)$. By (b), $u_j\in\operatorname{nul}(S-\sqrt{\lambda_j}I)$. Thus

$$Su_j=\sqrt{\lambda_j}u_j \quad \text{for all} \quad j\in\{1,\dots,n\}. \tag{8.2}$$

Let $v=\sum_{j=1}^{n}\langle v,u_j\rangle u_j$ be arbitrary vector in $\mathscr{V}$. Then, the linearity of S and (8.2) imply the claim in (c).

The implication (c) $\Rightarrow$ (a) is straightforward. $\square$

The implication (a) $\Rightarrow$ (c) of Theorem 8.3 yields that for a given nonnegative Q a nonnegative S such that $Q=S^2$ is uniquely determined. The common notation for this unique S is $\sqrt{Q}$.

Definition 8.4. Let $(\mathscr{V},\langle\,\cdot\,,\cdot\,\rangle)$ be a finite-dimensional inner product space over $\mathbb{C}$. An operator $U\in\mathscr{L}(\mathscr{V})$ is said to be **unitary** if $U^*U=I_{\mathscr{V}}$. $\lozenge$

Proposition 8.5. *Let $(\mathscr{V},\langle\,\cdot\,,\cdot\,\rangle)$ be a finite-dimensional inner product space over $\mathbb{C}$ and let $T\in\mathscr{L}(\mathscr{V})$. The following statements are equivalent.*

(a) *T is unitary.*
(b) *For all $u,v\in\mathscr{V}$ we have $\langle Tu,Tv\rangle=\langle u,v\rangle$.*
(c) *For all $v\in\mathscr{V}$ we have $\|Tv\|=\|v\|$.*
(d) *T is normal and all its eigenvalues have modulus* 1.

Theorem 8.6 (Polar Decomposition in $\mathscr{L}(\mathscr{V})$)**.** *Let $(\mathscr{V},\langle\,\cdot\,,\cdot\,\rangle)$ be a finite-dimensional inner product space over $\mathbb{C}$. For every $T\in\mathscr{L}(\mathscr{V})$ there exist a unitary operator U in $\mathscr{L}(\mathscr{V})$ and a unique nonnegative $Q\in\mathscr{L}(\mathscr{V})$ such that $T=UQ$; U is unique if and only if T is invertible.*

Proof. First, notice that the operator T^*T is nonnegative: for every $v\in\mathscr{V}$ we have

$$\langle T^*Tv,v\rangle=\langle Tv,Tv\rangle=\|Tv\|^2\geq 0.$$

To prove the uniqueness of Q assume that $T=UQ$ with U unitary and Q nonnegative. Then $Q^*=Q$, $U^*=U^{-1}$ and therefore, $T^*T=Q^*U^*UQ=QU^{-1}UQ=Q^2$. Since Q is nonnegative we have $Q=\sqrt{T^*T}$.

Set $Q=\sqrt{T^*T}$. By Theorem 8.3(b) we have $\operatorname{nul}Q=\operatorname{nul}(T^*T)$. Moreover, we have $\operatorname{nul}(T^*T)=\operatorname{nul}T$. The inclusion $\operatorname{nul}T\subseteq\operatorname{nul}(T^*T)$ is trivial. For the converse inclusion notice that $v\in\operatorname{nul}(T^*T)$ implies $T^*Tv=0$, which yields $\langle T^*Tv,v\rangle=0$ and thus $\langle Tv,Tv\rangle=0$. Consequently, $\|Tv\|=0$, that is $Tv=0$, yielding $v\in\operatorname{nul}T$. So,

$$\operatorname{nul}Q=\operatorname{nul}(T^*T)=\operatorname{nul}T \tag{8.3}$$

is proved.

First assume that T is invertible. By (8.3) and the Nullity-Rank Theorem, Q is invertible as well. Therefore $T=UQ$ is equivalent to $U=TQ^{-1}$ in this case. Since Q is unique, this proves the uniqueness of U. Set $U=TQ^{-1}$.

Since Q is self-adjoint, Q^{-1} is also self-adjoint. Therefore $U^* = Q^{-1}T^*$, yielding $U^*U = Q^{-1}T^*TQ^{-1} = Q^{-1}Q^2Q^{-1} = I_{\mathscr{V}}$. That is, U is unitary.

Now assume that T is not invertible. Since by (8.3) we have $\operatorname{nul} Q = \operatorname{nul} T$, the Nullity-Rank Theorem implies that $\dim(\operatorname{ran} Q) = \dim(\operatorname{ran} T)$. Notice that $\operatorname{nul} Q = (\operatorname{ran} Q)^{\perp}$ since Q is self-adjoint. Since T is not invertible, $\dim(\operatorname{ran} Q) = \dim(\operatorname{ran} T) < \dim \mathscr{V}$, implying that

$$\dim(\operatorname{nul} Q) = \dim\bigl((\operatorname{ran} Q)^{\perp}\bigr) = \dim\bigl((\operatorname{ran} T)^{\perp}\bigr) > 0. \tag{8.4}$$

We have two orthogonal decompositions of $\mathscr{V}$:

$$\mathscr{V} = (\operatorname{ran} Q) \oplus (\operatorname{nul} Q) = (\operatorname{ran} T) \oplus \bigl((\operatorname{ran} T)^{\perp}\bigr).$$

These two orthogonal decompositions are compatibile in the sense that the corresponding components have same dimensions, that is

$$\dim(\operatorname{ran} Q) = \dim(\operatorname{ran} T) \quad \text{and} \quad \dim(\operatorname{nul} Q) = \dim\bigl((\operatorname{ran} T)^{\perp}\bigr).$$

We will define $U : \mathscr{V} \to \mathscr{V}$ in two steps based on these two orthogonal decompositions. First we define the action of U on $\operatorname{ran} Q$, that is we define the operator $U_{\mathrm{r}} : \operatorname{ran} Q \to \operatorname{ran} T$, then we define an operator $U_{\mathrm{n}} : \operatorname{nul} Q \to (\operatorname{ran} T)^{\perp}$.

(A note on notation: Here I decided to call these operators U_{r} and U_{n} since the first is defined on the range, $\operatorname{ran} Q$, and the other on the null space, $\operatorname{nul} Q$, of Q. So, in the index, I just used the roman, not italic, first letter of their domain.)

We define $U_{\mathrm{r}} : \operatorname{ran} Q \to \operatorname{ran} T$ in the following way: Let $u \in \operatorname{ran} Q$ be arbitrary and let $x \in \mathscr{V}$ be such that $u = Qx$. Then we set

$$U_{\mathrm{r}} u = Tx.$$

First we need to show that U_{r} is well defined. Let $x_1, x_2 \in \mathscr{V}$ be such that $u = Qx_1 = Qx_2$. Then, $x_1 - x_2 \in \operatorname{nul} Q$. Since $\operatorname{nul} Q = \operatorname{nul} T$, we thus have $x_1 - x_2 \in \operatorname{nul} T$. Consequently, $Tx_1 = Tx_2$, that is U_{r} is well defined.

Next we prove that U_{r} is angle-preserving. Let $u_1, u_2 \in \operatorname{ran} Q$ be arbitrary and let $x_1, x_1 \in \mathscr{V}$ be such that $u_1 = Qx_1$ and $u_2 = Qx_2$ and calculate

$$\begin{aligned}
\langle U_{\mathrm{r}} u_1, U_{\mathrm{r}} u_2 \rangle &= \bigl\langle U_{\mathrm{r}}(Qx_1), U_{\mathrm{r}}(Qx_2) \bigr\rangle \\
\boxed{\text{by definition of } U_{\mathrm{r}}} \quad &= \langle Tx_1, Tx_2 \rangle \\
\boxed{\text{by definition of adjoint}} \quad &= \langle T^*Tx_1, x_2 \rangle \\
\boxed{\text{by definition of } Q} \quad &= \langle Q^2 x_1, x_2 \rangle \\
\boxed{\text{since } Q \text{ is self-adjoint}} \quad &= \langle Qx_1, Qx_2 \rangle \\
\boxed{\text{by definition of } x_1, x_2} \quad &= \langle u_1, u_2 \rangle
\end{aligned}$$

Thus $U_{\mathrm{r}} : \operatorname{ran} Q \to \operatorname{ran} T$ is angle-preserving.

Next we define an angle-preserving operator

$$U_{\mathrm{n}} : \operatorname{nul} Q \to (\operatorname{ran} T)^{\perp}.$$

By (8.4), we can set

$$m = \dim(\operatorname{nul} Q) = \dim\big((\operatorname{ran} T)^{\perp}\big) > 0.$$

Let $e_1, \ldots, e_m$ be an orthonormal basis on $\operatorname{nul} Q$ and let $f_1, \ldots, f_m$ be an orthonormal basis on $(\operatorname{ran} T)^{\perp}$. For arbitrary $w \in \operatorname{nul} Q$ define

$$U_{\mathrm{n}} w = U_{\mathrm{n}}\left(\sum_{j=1}^{m} \langle w, e_j\rangle e_j\right) := \sum_{j=1}^{m} \langle w, e_j\rangle f_j.$$

Then, for $w_1, w_2 \in \operatorname{nul} Q$ we have

$$\begin{aligned}
\langle U_{\mathrm{n}} w_1, U_{\mathrm{n}} w_2\rangle &= \left\langle \sum_{i=1}^{m} \langle w_1, e_i\rangle f_i, \sum_{j=1}^{m} \langle w_2, e_j\rangle f_j \right\rangle \\
&= \sum_{j=1}^{m} \langle w_1, e_j\rangle \overline{\langle w_2, e_j\rangle} \\
&= \langle w_1, w_2\rangle.
\end{aligned}$$

Hence U_{n} is angle-preserving on $(\operatorname{ran} Q)^{\perp}$.

Since the orthomormal bases in the definition of U_{n} were arbitrary and since $m > 0$, the operator U_{n} is not unique.

Finally we define $U : \mathscr{V} \to \mathscr{V}$ as a direct sum of U_{r} and U_{n}. Recall that

$$\mathscr{V} = (\operatorname{ran} Q) \oplus (\operatorname{nul} Q).$$

Let $v \in \mathscr{V}$ be arbitrary. Then there exist unique $u \in (\operatorname{ran} Q)$ and $w \in (\operatorname{nul} Q)$ such that $v = u + w$. Set

$$Uv = U_{\mathrm{r}} u + U_{\mathrm{n}} w.$$

We claim that U is angle-preserving. Let $v_1, v_2 \in \mathscr{V}$ be arbitrary and let $v_i = u_i + w_i$ with $u_i \in (\operatorname{ran} Q)$ and $w_i \in (\operatorname{nul} Q)$ with $i \in \{1, 2\}$. Notice that

$$\langle v_1, v_2\rangle = \langle u_1 + w_1, u_2 + w_2\rangle = \langle u_1, u_2\rangle + \langle w_1, w_2\rangle, \tag{8.5}$$

since u_1, u_2 are orthogonal to w_1, w_2. Similarly

$$\langle U_{\mathrm{r}} u_1 + U_{\mathrm{n}} w_1, U_{\mathrm{r}} u_2 + U_{\mathrm{n}} w_2\rangle = \langle U_{\mathrm{r}} u_1, U_{\mathrm{r}} u_2\rangle + \langle U_{\mathrm{n}} w_1, U_{\mathrm{n}} w_2\rangle, \tag{8.6}$$

since $U_{\mathrm{r}} u_1, U_{\mathrm{r}} u_2 \in (\operatorname{ran} T)$ and $U_{\mathrm{n}} w_1, U_{\mathrm{n}} w_2 \in (\operatorname{ran} T)^{\perp}$. Now we calculate, starting with the definition of U,

$$\begin{aligned}
\langle U v_1, U v_2\rangle &= \langle U_{\mathrm{r}} u_1 + U_{\mathrm{n}} w_1, U_{\mathrm{r}} u_2 + U_{\mathrm{n}} w_2\rangle \\
\boxed{\text{by (8.6)}} \quad &= \langle U_{\mathrm{r}} u_1, U_{\mathrm{r}} u_2\rangle + \langle U_{\mathrm{n}} w_1, U_{\mathrm{n}} w_2\rangle \\
\boxed{U_{\mathrm{r}} \text{ and } U_{\mathrm{n}} \text{ are angle-preserving}} \quad &= \langle u_1, u_2\rangle + \langle w_1, w_2\rangle \\
\boxed{\text{by (8.5)}} \quad &= \langle v_1, v_2\rangle.
\end{aligned}$$

Hence U is angle-preserving and by Proposition 8.5 we have that U is unitary.

Finally we show that $T = UQ$. Let $v \in \mathscr{V}$ be arbitrary. Then $Qv \in \operatorname{ran} Q$. By definitions of U and U_{r} we have

$$UQv = U_{\mathrm{r}}Qv = Tv.$$

Thus $T = UQ$, where U is unitary and Q is nonnegative. □

Corollary 8.7. *Let $(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle)$ be a finite-dimensional inner product space over $\mathbb{C}$. Let $T \in \mathscr{L}(\mathscr{V})$ and let $T = UQ$ be a polar decomposition of T. Then T is normal if and only if the unitary operator $U \in \mathscr{L}(\mathscr{V})$ and the nonnegative operator $Q \in \mathscr{L}(\mathscr{V})$ commute.*

9. Singular Value Decomposition

The following theorem is long. It deals with an arbitrary nonzero operator between finite-dimensional inner product spaces over $\mathbb{C}$. Its main parts are (I) and (IV). Part (I) establishes the existence of a Singular Value Decomposition, while in Part (IV), we prove the existence and uniqueness of the Moore-Penrose inverse for such an operator.

Theorem 9.1. *Let $m, n \in \mathbb{N}$. Let $(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{V}})$ and $(\mathscr{W}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{W}})$ be finite-dimensional inner product spaces over $\mathbb{C}$ such that $m = \dim \mathscr{V}$ and $n = \dim \mathscr{W}$. Let $T \in \mathscr{L}(\mathscr{V}, \mathscr{W})$ be a nonzero operator. Then there exist $r \in \mathbb{N}$ such that $r \leq \min\{m, n\}$, positive scalars $\sigma_1, \ldots, \sigma_r$ and orthonormal bases $\mathscr{B} = (v_1, \ldots, v_m)$ of $\mathscr{V}$ and $\mathscr{C} = (w_1, \ldots, w_n)$ of $\mathscr{W}$ such that the following statements hold:*

(I) *For every $v \in \mathscr{V}$ we have*

$$Tv = \sum_{j=1}^{r} \sigma_j \langle v, v_j\rangle_{\mathscr{V}}\, w_j. \tag{9.1}$$

(II) *The top left block corner of the $n \times m$ matrix $M_{\mathscr{C}}^{\mathscr{B}}(T)$, which is of size $r \times r$, is a diagonal matrix with positive diagonal entries $\sigma_1, \ldots, \sigma_r$. All other entries of $M_{\mathscr{C}}^{\mathscr{B}}(T)$ are equal to 0. That is,*

$$M_{\mathscr{C}}^{\mathscr{B}}(T) = \begin{array}{r} r\left\{\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right. \\ n-r\left\{\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right. \end{array} \begin{array}{c}\left[\begin{array}{ccc|ccc} \sigma_1 & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & \cdots & \sigma_r & 0 & \cdots & 0\\ \hline 0 & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{array}\right] \\ \underbrace{\hphantom{\sigma_1\ \cdots\ \sigma_r}}_{r}\ \underbrace{\hphantom{0\ \cdots\ 0\ 0}}_{m-r} \end{array}$$

$n \times m$ matrix with a diagonal $r \times r$ top left block and with all other entries equal to 0.

Or, in block-matrix notation

$$M_{\mathscr{C}}^{\mathscr{B}}(T) = \left[\begin{array}{c|c} \Sigma_r & 0 \\ \hline 0 & 0 \end{array}\right], \qquad (n \times m \ \ \text{matrix})$$

where Σ_r is an $r \times r$ diagonal matrix with positive entries $\sigma_1, \dots, \sigma_r$ on the diagonal and the zero matrices of the appropriate sizes.

(III) *For every $w \in \mathscr{W}$ we have*

$$T^* w = \sum_{j=1}^{r} \sigma_j \langle w, w_j \rangle_{\mathscr{W}} v_j. \tag{9.2}$$

Equivalently,

$$M_{\mathscr{B}}^{\mathscr{C}}(T^*) = \left[\begin{array}{c|c} \Sigma_r & 0 \\ \hline 0 & 0 \end{array}\right], \qquad (m \times n \quad \textit{matrix})$$

(IV) *Let $S \in \mathscr{L}(\mathscr{W}, \mathscr{V})$. The following three statements are equivalent.*

(i) *S satisfies the following four equations*

$$TST = T, \quad STS = S, \quad (TS)^* = TS, \quad (ST)^* = ST, \tag{9.3}$$

(ii) *For every $w \in \mathscr{W}$ we have*

$$Sw = \sum_{j=1}^{r} \frac{1}{\sigma_j} \langle w, w_j \rangle_{\mathscr{W}} v_j. \tag{9.4}$$

(iii)

$$M_{\mathscr{B}}^{\mathscr{C}}(S) = \left[\begin{array}{c|c} \Sigma_r^{-1} & 0 \\ \hline 0 & 0 \end{array}\right]. \qquad (m \times n \quad \textit{matrix})$$

Proof. (I) Let $T^* \in \mathscr{L}(\mathscr{W}, \mathscr{V})$ be the adjoint of T. Since for all $v \in \mathscr{V}$ we have

$$\langle T^*Tv, v \rangle_{\mathscr{V}} = \langle Tv, Tv \rangle_{\mathscr{W}} \geq 0,$$

the operator $T^*T \in \mathscr{L}(\mathscr{V})$ is nonnegative, and, as such, self-adjoint with the nonnegative eigenvalues $\lambda_1, \dots, \lambda_n$. We assume that the eigenvalues are ordered in nonincreasing order $\lambda_1 \geq \cdots \geq \lambda_n$. Since $T \neq 0_{\mathscr{L}(\mathscr{V})}$ we have $\lambda_1 > 0$. Set

$$r = \max\bigl\{k \in \{1, \dots, n\} : \lambda_k > 0\bigr\}. \tag{9.5}$$

Thus, for all $k \in \{1, \dots, n\}$, if $k \leq r$, then $\lambda_k > 0$, and, if $k > r$, then $\lambda_k = 0$. Set

$$\sigma_k = \sqrt{\lambda_k}, \qquad k \in \{1, \dots, r\}. \tag{9.6}$$

Since T^*T is a self-adjoint operator on $\mathscr{V}$, there exists an orthonormal basis $\mathscr{B} = (v_1, \dots, v_m)$ of $\mathscr{V}$ such that

$$\forall k \in \{1, \dots, n\} \quad T^*Tv_k = \lambda_k v_k. \tag{9.7}$$

Recall that

$$\operatorname{nul}(T) = \operatorname{nul}(T^*T) \quad \text{and} \quad \operatorname{ran}(T^*) = \operatorname{ran}(T^*T)$$

It follows from the definition of r in (9.5) and (9.7) that

$$\operatorname{nul}(T) = \operatorname{nul}(T^*T) = \operatorname{span}\bigl\{v_k : k \in \{1, \dots, n\} \wedge k > r\bigr\}.$$

Since T^*T is self-adjoint and since $\mathscr{B}$ is an orthonormal basis of $\mathscr{V}$, (9.5) and (9.7) imply

$$\operatorname{ran}(T^*) = \operatorname{ran}(T^*T) = \bigl(\operatorname{nul}(T^*T)\bigr)^{\perp} = \operatorname{span}\bigl\{v_k : k \in \{1,\ldots,n\} \wedge k \leq r\bigr\}.$$

Therefore

$$r = \dim \operatorname{ran}(T^*).$$

Notice that for all $k \in \{1,\ldots,r\}$ we have

$$0 < \lambda_k = (\sigma_k)^2 = \lambda_k \langle v_k, v_k\rangle_{\mathscr{V}} = \langle T^*Tv_k, v_k\rangle_{\mathscr{V}} = \langle Tv_k, Tv_k\rangle_{\mathscr{W}} = \|Tv_k\|_{\mathscr{W}}^2,$$

and define r unit vectors in $\operatorname{ran}(T) \subseteq \mathscr{W}$ as follows

$$w_k = \frac{1}{\sigma_k} T v_k, \qquad k \in \{1,\ldots,r\}.$$

The following calculation shows that the vectors $w_1,\ldots,w_r$ are mutually orthogonal. Let $j,k \in \{1,\ldots,r\}$ be arbitrary and such that $j \neq k$. Then

$$\langle w_j, w_k\rangle_{\mathscr{W}} = \frac{1}{\sigma_j\sigma_k}\langle Tv_j, Tv_k\rangle_{\mathscr{W}} = \frac{1}{\sigma_j\sigma_k}\langle T^*Tv_j, v_k\rangle_{\mathscr{V}} = \frac{\lambda_j}{\sigma_j\sigma_k}\langle v_j, v_k\rangle_{\mathscr{V}} = 0,$$

since $\mathscr{B}$ is an orhonormal basis for $\mathscr{V}$. Consequently, $w_1,\ldots,w_r$ are linearly independent in $\mathscr{W}$. Hence, $r \leq \min\{m,n\}$.

Since

$$r + \dim \operatorname{nul}(T) = m = \dim \mathscr{V}$$

and, by the Nullity-Rank Theorem,

$$\dim \operatorname{nul}(T) + \dim \operatorname{ran}(T) = m = \dim \mathscr{V},$$

we deduce that $r = \dim \operatorname{ran}(T)$. Hence $\{w_1,\ldots,w_r\}$ is an orthonormal basis for $\operatorname{ran}(T)$. If $\operatorname{ran}(T)$ is a proper subspace of $\mathscr{W}$, since $\bigl(\operatorname{ran}(T)\bigr)^{\perp} = \operatorname{nul}(T^*)$, choosing $w_{r+1},\ldots,w_n$ to be an orthonormal basis for $\operatorname{nul}(T^*)$ we obtain an orthonormal basis $\mathscr{C} = \{w_1,\ldots,w_n\}$ for $\mathscr{W}$. Let $v \in \mathscr{V}$ be arbitrary and calculate

$$\begin{aligned}
Tv &= T\left(\sum_{k=1}^{m} \langle v, v_k\rangle_{\mathscr{V}} v_k\right) \\
\boxed{\text{linearity of } T} \quad &= \sum_{k=1}^{m} \langle v, v_k\rangle_{\mathscr{V}} T v_k \\
\boxed{\text{definition of } r} \quad &= \sum_{k=1}^{r} \langle v, v_k\rangle_{\mathscr{V}} T v_k \\
\boxed{\text{definition of } w_k} \quad &= \sum_{k=1}^{r} \langle v, v_k\rangle_{\mathscr{V}} \sigma_k w_k \\
&= \sum_{k=1}^{r} \sigma_k \langle v, v_k\rangle_{\mathscr{V}} w_k.
\end{aligned}$$

(III) Define $S \in \mathscr{L}(\mathscr{W}, \mathscr{V})$ by: For every $w \in \mathscr{W}$ set

$$Sw = \sum_{j=1}^{r} \sigma_j \langle w, w_j \rangle_{\mathscr{W}} v_j.$$

For an arbitrary $v \in \mathscr{V}$ and an arbitrary $w \in \mathscr{W}$ calculate

$$\begin{aligned}
\langle v, Sw \rangle_{\mathscr{V}} &= \left\langle v, \sum_{j=1}^{r} \sigma_j \langle w, w_j \rangle_{\mathscr{W}} v_j \right\rangle_{\mathscr{V}} \\
&= \sum_{j=1}^{r} \sigma_j \overline{\langle w, w_j \rangle_{\mathscr{W}}} \langle v, v_j \rangle_{\mathscr{V}} \\
&= \sum_{j=1}^{r} \sigma_j \langle v, v_j \rangle_{\mathscr{V}} \langle w_j, w \rangle_{\mathscr{W}} \\
&= \left\langle \sum_{j=1}^{r} \sigma_j \langle v, v_j \rangle_{\mathscr{V}} w_j, w \right\rangle_{\mathscr{W}} \\
&= \langle Tv, w \rangle_{\mathscr{W}}.
\end{aligned}$$

Since $v \in \mathscr{V}$ and $w \in \mathscr{W}$ were arbitrary, the preceding calculation proves that $S = T^*$. (citation)

(IV) The equivalence (ii)$\Leftrightarrow$(iii) follows from the definition of the matrices Σ_r and $M_{\mathscr{B}}^{\mathscr{C}}(S)$.

To prove (ii)$\Rightarrow$(i), assume (ii). (This is proof of the existence of the Moore-Penrose inverse.) Then, (9.2) and (9.4) imply that $\operatorname{ran}(S) = \operatorname{ran}(T^*)$ and $\operatorname{nul}(S) = \operatorname{nul}(T^*)$. Further, (9.1) and (9.4) yield

$$TS = P_{\operatorname{ran}(T)} = P_{\operatorname{nul}(S)^{\perp}} \quad \text{and} \quad ST = P_{\operatorname{ran}(S)} = P_{\operatorname{nul}(T)^{\perp}}.$$

Since an orthogonal projection is a self-adjoint operator, both TS and ST are self-adjoint. Since $P_{\operatorname{ran}(T)}T = T$ and $P_{\operatorname{ran}(S)}S = S$, we deduce that $TST = T$ and $STS = S$. Thus (ii)$\Rightarrow$(i).

To prove (i)$\Rightarrow$(iii), assume (i). (This is proof of the uniqueness of the Moore-Penrose inverse.) Let

$$M_{\mathscr{B}}^{\mathscr{C}}(S) = \left[\begin{array}{c|c} \mathsf{A} & \mathsf{B} \\ \hline \mathsf{C} & \mathsf{D} \end{array}\right], \qquad (m \times n \ \text{ matrix})$$

where A is an $r \times r$ matrix, B is $r \times (n-r)$ matrix, C is $(m-r) \times r$ matrix, and D is $(m-r) \times (n-r)$ matrix. We proved in (II)

$$M_{\mathscr{C}}^{\mathscr{B}}(T) = \left[\begin{array}{c|c} \Sigma_r & 0 \\ \hline 0 & 0 \end{array}\right], \qquad (n \times m \ \text{ matrix})$$

with Σ_r being an $r \times r$ diagonal matrix with positive entries on the diagonal and the zeros of appropriate sizes. Then

$$M_{\mathscr{C}}^{\mathscr{C}}(TS) = \left[\begin{array}{c|c} \Sigma_r \mathsf{A} & \Sigma_r \mathsf{B} \\ \hline \mathsf{0} & \mathsf{0} \end{array}\right], \qquad (n \times n \ \ \text{matrix})$$

and

$$M_{\mathscr{B}}^{\mathscr{B}}(ST) = \left[\begin{array}{c|c} \mathsf{A}\Sigma_r & \mathsf{0} \\ \hline \mathsf{C}\Sigma_r & \mathsf{0} \end{array}\right]. \qquad (m \times m \ \ \text{matrix})$$

Since TS and ST are self-adjoint, we deduce that $\Sigma_r B = 0$ and $C\Sigma_r = 0$. Consequently, $B = 0$ and $C = 0$ as Σ_r is invertible. Since $TST = T$, the operator TS acts as an identity on $\operatorname{ran}(T)$. Therefore $\Sigma_r A = I_r$. Hence $A = \Sigma_r^{-1}$. Hence,

$$M_{\mathscr{B}}^{\mathscr{C}}(S) = \left[\begin{array}{c|c} \Sigma_r^{-1} & \mathsf{0} \\ \hline \mathsf{0} & \mathsf{D} \end{array}\right].$$

Now the equality $S = STS$ yields

$$\begin{aligned} \left[\begin{array}{c|c} \Sigma_r^{-1} & \mathsf{0} \\ \hline \mathsf{0} & \mathsf{D} \end{array}\right] &= M_{\mathscr{B}}^{\mathscr{C}}(S) \\ &= M_{\mathscr{B}}^{\mathscr{C}}(STS) \\ &= M_{\mathscr{B}}^{\mathscr{C}}(S) M_{\mathscr{C}}^{\mathscr{B}}(T) M_{\mathscr{B}}^{\mathscr{C}}(S) \\ &= \left[\begin{array}{c|c} \Sigma_r^{-1} & \mathsf{0} \\ \hline \mathsf{0} & \mathsf{D} \end{array}\right] \left[\begin{array}{c|c} \Sigma_r & \mathsf{0} \\ \hline \mathsf{0} & \mathsf{0} \end{array}\right] \left[\begin{array}{c|c} \Sigma_r^{-1} & \mathsf{0} \\ \hline \mathsf{0} & \mathsf{D} \end{array}\right] \\ &= \left[\begin{array}{c|c} \Sigma_r^{-1} & \mathsf{0} \\ \hline \mathsf{0} & \mathsf{D} \end{array}\right] \left[\begin{array}{c|c} \mathsf{I}_r & \mathsf{0} \\ \hline \mathsf{0} & \mathsf{0} \end{array}\right] \\ &= \left[\begin{array}{c|c} \Sigma_r^{-1} & \mathsf{0} \\ \hline \mathsf{0} & \mathsf{0} \end{array}\right]. \end{aligned}$$

Hence, $\mathsf{D} = \mathsf{0}$, and consequently,

$$M_{\mathscr{B}}^{\mathscr{C}}(S) = \left[\begin{array}{c|c} \Sigma_r^{-1} & \mathsf{0} \\ \hline \mathsf{0} & \mathsf{0} \end{array}\right].$$

This proves (i)⇒(iii). Since we proved

$$\text{(i)} \Rightarrow \text{(iii)} \Rightarrow \text{(ii)} \Rightarrow \text{(i)}$$

proof of (IV) is complete. □

Definition 9.2. The values $\sigma_1, \ldots, \sigma_r$ from Theorem 9.1, which are in fact the square-roots of the positive eigenvalues of T^*T, are called the **singular values** of T. Equality (9.1) or the matrix in (II) is called a **singular value decomposition** of T. ◊

Definition 9.3. For $T \in \mathscr{L}(\mathscr{V}, \mathscr{W})$, the unique operator $T^+ \in \mathscr{L}(\mathscr{W}, \mathscr{V})$ that satisfies the equalities

$$TT^+T = T, \quad T^+TT^+ = T^+, \quad (TT^+)^* = TT^+, \quad (T^+T)^* = T^+T. \tag{9.8}$$

is called the **Moore-Penrose inverse** of T, $\Diamond$

10. Problems

Exercise 10.1. Let $\mathscr{V}$ be a vector space over $\mathbb{F}$ with a positive definite inner product $\langle\,\cdot\,,\cdot\,\rangle$ and let $\mathscr{U}$ be a subspace of $\mathscr{V}$. Prove that $\bigl((\mathscr{U}^\perp)^\perp\bigr)^\perp = \mathscr{U}^\perp$. $\Diamond$