Complex Analysis and Operator Theory (2025) 19:201
https://doi.org/10.1007/s11785-025-01812-5

**Complex Analysis and Operator Theory**

# Operators Without Eigenvalues in Finite-Dimensional Vector Spaces: Self-Adjoint Couplings and Shtraus Subspaces

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Received: 21 June 2025 / Accepted: 10 September 2025


**Abstract**

In a coupling theorem from 2001 we described a special class of canonical self-adjoint extensions of the direct sum of symmetric linear relations $S_1$ and $S_2$ in Krein spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ and assigned a unique parameter to each of these extensions. In this paper we assume that $\dim \mathfrak{H}_2 \in \mathbb{N}$ and that $S_2$ is an operator without eigenvalues and construct a model for $(\mathfrak{H}_2, S_2)$ based on an essentially unique polynomial matrix $\mathcal{P}(z)$. The families of Shtraus subspaces associated with the self-adjoint extensions are characterized as restrictions of $S_1^*$ by polynomial boundary conditions involving $\mathcal{P}(z)$ and the parameters. We establish necessary and sufficient conditions on the parameters under which the extensions are similar and the corresponding families of Shtraus subspaces coincide. Related to our results is the equation $\mathcal{W}(z)\mathcal{P}(z) = \mathcal{P}(z)\mathsf{V}$ in which the unimodular matrix polynomial $\mathcal{W}(z)$ and the invertible matrix $\mathsf{V}$ are the unknowns. Explicit examples are given.

**Keywords** Matrix polynomial · Krein space · Pontryagin space · Reproducing kernel · Symmetric linear relation · Boundary mapping · Nilpotent operator · Self-adjoint extension · Canonical space of vector polynomials

**Mathematics Subject Classification** 15A03 · 46C20 · 46E22 · 47A06 · 47A45

Communicated by Vladimir Bolotnikov.

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Published online: 28 September 2025

## 1 Introduction

The starting point of this paper is the Coupling Theorem [12, Theorem 5.1] which we recall as Theorem 3.1 in Section 3. It describes the canonical self-adjoint extensions $\widetilde{A}$ of the direct sum of two closed symmetric linear relations $S_1$ in the Krein space $\mathfrak{H}_1$ and $S_2$ in the Krein space $\mathfrak{H}_2$ such that $\widetilde{A} \cap \mathfrak{H}_1^2 = S_1$ and $\widetilde{A} \cap \mathfrak{H}_2^2 = S_2$. We assume that the symmetric linear relations have the same equal defect numbers $d \in \mathbb{N}$. If we fix boundary mappings $\mathsf{b}_1$ for $S_1$ and $\mathsf{b}_2$ for $S_2$, then all such extensions $\widetilde{A}$ are parametrized by invertible $2d \times 2d$ matrices $\Gamma$ satisfying (3.3). To indicate this connection we sometimes write $\widetilde{A}_\Gamma$ for $\widetilde{A}$.

For such an $\widetilde{A}$ we consider the family of Shtraus subspaces $T_{\widetilde{A}}(z)$, $z \in \mathbb{C} \cup \{\infty\}$, in $\mathfrak{H}_1$, see Subsection 3.2. Each member $T_{\widetilde{A}}(z)$ of the family is a kind of compression of $\widetilde{A}$ to $\mathfrak{H}_1$ and satisfies $S_1 \subset T_{\widetilde{A}}(z) \subset S_1^*$.

We assume that $\dim \mathfrak{H}_2 \in \mathbb{N}$ and that $S_2$ is an operator in $\mathfrak{H}_2$ without eigenvalues. Then the Model Theorem 4.2 assigns an essentially unique $d \times 2d$ matrix polynomial $\mathcal{P}(z)$ to the pair $(\mathfrak{H}_2, S_2)$ with properties described in Theorem 2.4. We show that, as a consequence, $T_{\widetilde{A}_\Gamma}(z)$ can be characterized as the restriction of $S_1^*$ determined by the polynomial boundary condition $\mathcal{P}(z)\Gamma^{-1}\mathsf{b}_1(\{f_1, g_1\}) = 0$, $\{f_1, g_1\} \in S_1^*$. This is formulated in Theorem 4.4, our first main theorem.

The bijective correspondence between the class of self-adjoint extensions $\widetilde{A}_\Gamma$ and the class of invertible $2d \times 2d$ matrices $\Gamma$ satisfying (3.3) asserted in the Coupling Theorem cannot be carried over to a bijective correspondence between the Shtraus families $T_{\widetilde{A}_\Gamma}(z)$ and the parameters $\Gamma$. In this paper we study the correspondence in detail. This is summarized in Theorem 4.5, the second main theorem of the paper. We give necessary and sufficient conditions on $\Gamma$ and $\Lambda$ for which the identity $T_{\widetilde{A}_\Gamma}(z) = T_{\widetilde{A}_\Lambda}(z)$ holds for all $z \in \mathbb{C} \cup \{\infty\}$. We also show that this identity holds if and only if the relations $\widetilde{A}_\Gamma$ and $\widetilde{A}_\Lambda$ are similar under an isomorphism which leaves $\mathfrak{H}_1$ invariant. A key role in our results is played by the equation $\mathcal{W}(z)\mathcal{P}(z) = \mathcal{P}(z)\mathsf{V}$, $z \in \mathbb{C}$. Here $\mathcal{W}(z)$, a unimodular $d \times d$ matrix polynomial, and $\mathsf{V}$, an invertible $2d \times 2d$ matrix, are the unknowns. We provide explicit examples, obtained using Wolfram Mathematica, to illustrate that equations may have more than the trivial solution $\mathcal{W}(z) = a\mathsf{I}_d$ and $\mathsf{V} = a\mathsf{I}_{2d}$ with $a \in \mathbb{C}\backslash\{0\}$.

In a sequel to this paper we drop the assumption that $S_1 = \widetilde{A} \cap \mathfrak{H}_1^2$ in the Coupling Theorem, but still assume that $\dim \mathfrak{H}_2 \in \mathbb{N}$ and $S_2 = \widetilde{A} \cap \mathfrak{H}_2^2$ is an operator without eigenvalues. We show that the family of Shtraus subspaces $T_{\widetilde{A}}(z)$ for $\widetilde{A}$ can be described as the restriction of $S_1^*$ determined by a polynomial boundary condition of the form $\mathcal{P}(z)\mathsf{b}_1(\{f_1, g_1\}) = 0$, $\{f_1, g_1\} \in S_1^*$, where $\mathcal{P}(z)$ satisfies all the assumptions of Theorem 2.4 except assumption (d), that is, $\mathcal{P}(z)$ may have constant rows. We investigate the relation between the resolvent set, the point spectrum, the Jordan chains and the continuous spectrum of $\widetilde{A}$ and the same notions for the boundary eigenvalue problem

$\mathrm{BEP}\big(S_1, \mathsf{b}_1, \mathcal{P}(z)\big)$: *For all $\lambda \in \mathbb{C}$ and all $h_1 \in \mathfrak{H}_1$ determine the existence and the uniqueness of a solution $\{f_1, g_1\} \in \mathfrak{H}_1^2$ of the system*

$$\{f_1, g_1\} \in S_1^*, \quad g_1 - \lambda f_1 = h_1 \quad \text{and} \quad \mathcal{P}(\lambda)\mathsf{b}_1(\{f_1, g_1\}) = 0.$$

*For* $\lambda = \infty$ *and all* $h_1 \in \mathfrak{H}_1$ *determine the existence and uniqueness of a solution* $\{h_1, f_1\} \in \mathfrak{H}_1^2$ *of the system*

$$\{h_1, f_1\} \in S_1^* \quad \text{and} \quad \mathsf{P}_\infty \mathsf{b}_1(\{h_1, f_1\}) = 0.$$

The latter system corresponds to a canonical self-adjoint extension of $S_1$ described in Theorem 2.3.

References to earlier results related to the results in this paper are given in the relevant sections. We assume that the reader is familiar with indefinite metric spaces, linear relations defined on them such as symmetric and self-adjoint ones, and reproducing kernel spaces.

### 1.1 Notation and Preliminaries

By $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ we denote the sets of positive integers, real numbers and complex numbers, and $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the one point compactification of $\mathbb{C}$. For $z \in \overline{\mathbb{C}}$, $z^* \in \overline{\mathbb{C}}$ denotes the complex conjugate of $z$; we have $\infty^* = \infty$. The asterisk as a superscript is also used to denote the adjoint of a matrix, operator or linear relation. For $m, n \in \mathbb{N}$ we denote by $\mathsf{I}_n$ the $n \times n$ identity matrix, while $\mathsf{0}_{m\times n}$ denotes the $m \times n$ zero matrix and $\mathsf{0}$ stands for the zero matrix whose size is implied by the context. For constant matrices we use the sans-serif font, except for matrices $\Gamma$ and $\Lambda$ which we use exclusively in boundary conditions. For matrix polynomials we use the calligraphic font accompanied with the variable. $\#S$ means the number of elements in a finite set $S$.

In a vector space $\mathfrak{H}$ we often identify an operator in $\mathfrak{H}$ with its graph in

$$\mathfrak{H}^2 = \big\{\{u, v\} \,:\, u, v \in \mathfrak{H}\big\}$$

and then we denote them by the same symbol. Here the notation $\{u, v\}$ stands for the ordered pair; although we use curly brackets also to denote sets, the meaning will be clear from the context. For example, the scaled identity operator $\alpha I$ on $\mathfrak{H}$, with $\alpha \in \mathbb{C}$, is identified with $\alpha I = \big\{\{u, \alpha u\} \,:\, u \in \mathfrak{H}\big\}$. A *linear relation* $S$ in $\mathfrak{H}$ is a vector subspace $S$ of $\mathfrak{H}^2$. It is the graph of an operator $S$ if and only if $\{0, v\} \in S$ implies $v = 0$, and then we write $v = Su$ for $\{u, v\} \in S$. For a linear relation $S$ in $\mathfrak{H}$ we define the *domain* $\operatorname{dom} S$, the *range* $\operatorname{ran} S$, the *null space* $\operatorname{nul} S$, and the *multivalued part* $\operatorname{mul} S$ of $S$ as follows

$$\operatorname{dom} S = \big\{u \in \mathfrak{H} : \{u, v\} \in S \text{ with } v \in \mathfrak{H}\big\}, \qquad \operatorname{nul} S = \big\{u \in \mathfrak{H} : \{u, 0\} \in S\big\},$$
$$\operatorname{ran} S = \big\{v \in \mathfrak{H} : \{u, v\} \in S \text{ with } u \in \mathfrak{H}\big\}, \qquad \operatorname{mul} S = \big\{v \in \mathfrak{H} : \{0, v\} \in S\big\}.$$

The *sum* $S + T$, the *difference* $S - T$, and the *product* $TS$ of two linear relations $S$ and $T$ in $\mathfrak{H}$ are defined by

$$S \pm T = \big\{\{u, v \pm w\} : \{u, v\} \in S,\ \{u, w\} \in T\big\},$$

and

$$TS = \big\{\{u, w\} : \{u, v\} \in S,\ \{v, w\} \in T \text{ with } v \in \mathfrak{H}\big\}.$$

For example, $\alpha S = (\alpha I)S = \big\{\{u, \alpha v\} : \{u, v\} \in S\big\}$. Since $S - \alpha = S - \alpha I = \big\{\{u, v - \alpha u\} : \{u, v\} \in S\big\}$, we have $\operatorname{nul}(S - \alpha) = \operatorname{dom}(S \cap \alpha I)$. The product of linear relations is associative. For a linear relation $S$ in $\mathfrak{H}$ its nonnegative powers are defined inductively, with $S^0 = I$ and $S^1 = S$. A linear relation $S$ is said to be *nilpotent* if there exists an $m \in \mathbb{N}$ such that $S^m = 0$. If $S^{m-1} \neq 0$ and $S^m = 0$, then $m$ is the *nilpotency index* of $S$. Here the symbol 0 may denote either the *zero relation* $\big\{\{0, 0\}\big\} \subset \mathfrak{H}^2$, or the *zero operator* on $\mathfrak{H}$, that is, the relation $\big\{\{u, 0\} : u \in \mathfrak{H}\big\}$.

For $m, n \in \mathbb{N}$ we denote by $\mathbb{C}^{m\times n}[z]$ the space of all matrix polynomials with coefficients in $\mathbb{C}^{m\times n}$. The *degree of such a polynomial* is $-\infty$ if it is the zero polynomial, otherwise it is the highest power of $z$ for which the corresponding matrix coefficient is nonzero. A square matrix polynomial is called *unimodular* if its determinant is a nonzero scalar. We write $\mathbb{C}^m[z]$ for $\mathbb{C}^{m\times 1}[z]$. For vector functions $a(z)$ and $b(z)$, the identity $a(z) \equiv b(z)$ stands for the proposition $a(z) = b(z)$ for all $z \in \mathbb{C}$.

Let $\mathcal{S}(z) \in \mathbb{C}^{m\times n}[z]$ be a nonzero matrix polynomial of degree $s$; thus $s \in \{0\} \cup \mathbb{N}$. If

$$\mathcal{S}(z) = \mathsf{S}_0 + \mathsf{S}_1 z + \cdots + \mathsf{S}_s z^s, \quad \mathsf{S}_j \in \mathbb{C}^{m\times n}, \quad j \in \{0, \ldots, s\} \quad \text{with} \quad \mathsf{S}_s \neq 0$$

is the expansion of $\mathcal{S}(z)$ in powers of $z$ and $\mathsf{C}_{\mathcal{S}}$ stands for its $(s+1)m \times n$ *coefficient matrix*:

$$\mathsf{C}_{\mathcal{S}} = \begin{bmatrix} \mathsf{S}_0 \\ \mathsf{S}_1 \\ \vdots \\ \mathsf{S}_s \end{bmatrix},$$

then

$$\bigcap_{z\in\mathbb{C}} \operatorname{nul} \mathcal{S}(z) = \operatorname{nul} \mathsf{C}_{\mathcal{S}}. \tag{1.1}$$

Here nul denotes the null space of a matrix.

Assume that the rows of $\mathcal{S}(z)$ are nonzero and for $k \in \{1, \ldots, m\}$ denote the degree of the $k$-th row of $\mathcal{S}(z)$ by $\sigma_k$. Then $\sigma_k \in \{0\} \cup \mathbb{N}$. We define the $m \times n$ matrix $\mathsf{S}_\infty$ to be the matrix consisting of the leading coefficients of the rows of $\mathcal{S}(z)$. Specifically,

$$\mathsf{S}_\infty = \lim_{z\to\infty} \operatorname{diag}\big(z^{-\sigma_1}, \ldots, z^{-\sigma_m}\big)\mathcal{S}(z). \tag{1.2}$$

When convenient, we will extend $\mathcal{S}(z)$ to $\overline{\mathbb{C}}$ by defining $\mathcal{S}(\infty) = \mathsf{S}_\infty$.

A *Krein space* is an ordered pair $(\mathfrak{H}, [\cdot,\cdot]_{\mathfrak{H}})$ consisting of a complex vector space $\mathfrak{H}$ and a Hermitian form $[\cdot,\cdot]_{\mathfrak{H}}$ defined on $\mathfrak{H} \times \mathfrak{H}$, provided that there exists a linear involution (or self-inverse function) $J : \mathfrak{H} \to \mathfrak{H}$ such that $[J\cdot,\cdot]_{\mathfrak{H}}$ is a Hilbert space inner product on $\mathfrak{H}$. Such an involution $J$ is in fact a self-adjoint operator on this Hilbert space and it is called a *fundamental symmetry* on $(\mathfrak{H}, [\cdot,\cdot]_{\mathfrak{H}})$. Its spectral subspaces $\mathfrak{H}_-$ (corresponding to the eigenvalue $-1$) and $\mathfrak{H}_+$ (corresponding to the eigenvalue 1) form an orthogonal decomposition of $\mathfrak{H}$ both as a Hilbert space and as a Krein space. Moreover, the spaces $(\mathfrak{H}_\pm, \pm[\cdot,\cdot]_{\mathfrak{H}})$ are Hilbert spaces. Such a decomposition is called a *fundamental decomposition* of a Krein space. The dimensions of $\mathfrak{H}_-$ and $\mathfrak{H}_+$ are called the *negative* and *positive indices* of $\mathfrak{H}$, respectively. The *antispace* of a Krein space $(\mathfrak{H}, [\cdot,\cdot]_{\mathfrak{H}})$ is the Krein space $(\mathfrak{H}, -[\cdot,\cdot]_{\mathfrak{H}})$. A Krein space is called a *Pontryagin space* if either its positive or negative index is finite. A detailed study of Krein spaces and operators in them can be found in [4, 6, 27].

Let $(\mathfrak{G}, [\cdot,\cdot]_{\mathfrak{G}})$ and $(\mathfrak{H}, [\cdot,\cdot]_{\mathfrak{H}})$ be Krein spaces. By $T : \mathfrak{G} \to \mathfrak{H}$ is an *isomorphism* we mean that $T$ is a linear bijection and isometry between the Krein spaces, that is, for all $u, v \in \mathfrak{G}$ we have $[Tu, Tv]_{\mathfrak{H}} = [u, v]_{\mathfrak{G}}$. In this case $T^*T = I_{\mathfrak{G}}$ and $T^* = T^{-1}$.

## 2 The Polynomial $\mathcal{P}(z)$ and Canonical Subspaces

### 2.1 Boundary Mapping and Gram Matrix

We collect basic facts about defect numbers of symmetric linear relations in Krein spaces, boundary mappings and Gram matrices.

Let $(\mathfrak{H}, [\cdot,\cdot]_{\mathfrak{H}})$ be a Krein space and let $J$ be a fundamental symmetry on it. Then $\mathfrak{H}$, equipped with the inner product $\langle\cdot,\cdot\rangle_{\mathfrak{H}} = [J\cdot,\cdot]_{\mathfrak{H}}$, is a Hilbert space.

The Cartesian product $\mathfrak{H}^2$ endowed with the indefinite inner product

$$\langle\!\langle \{f, g\}, \{h, k\} \rangle\!\rangle = \mathrm{i}\big(\langle f, k\rangle_{\mathfrak{H}} - \langle g, h\rangle_{\mathfrak{H}}\big) \quad \text{for all} \quad \{f, g\}, \{h, k\} \in \mathfrak{H}^2$$

is a Krein space. For example, the direct sum $\mathfrak{H}^2 = (-\mathrm{i}I_{\mathfrak{H}}) \oplus (\mathrm{i}I_{\mathfrak{H}})$ of the graphs of the operators $-\mathrm{i}I_{\mathfrak{H}}$ and $\mathrm{i}I_{\mathfrak{H}}$ is a fundamental decomposition of the Krein space $(\mathfrak{H}^2, \langle\!\langle\cdot,\cdot\rangle\!\rangle)$.

Furthermore, $\mathfrak{H}^2$ equipped with the inner product

$$[\![\{f, g\}, \{h, k\}]\!] = \mathrm{i}\big([f, k]_{\mathfrak{H}} - [g, h]_{\mathfrak{H}}\big) \quad \text{for all} \quad \{f, g\}, \{h, k\} \in \mathfrak{H}^2 \tag{2.1}$$

is a Krein space as well, since it is straightforward to verify that the mapping $\{f, g\} \mapsto \{f, Jg\}$ is an isomorphism between the inner product spaces $(\mathfrak{H}^2, [\![\cdot,\cdot]\!])$ and $(\mathfrak{H}^2, \langle\!\langle\cdot,\cdot\rangle\!\rangle)$.

The idea of using the Krein spaces $(\mathfrak{H}^2, [\![\cdot,\cdot]\!])$ and $(\mathfrak{H}^2, \langle\!\langle\cdot,\cdot\rangle\!\rangle)$ to study self-adjoint extensions of symmetric relations in $(\mathfrak{H}, [\cdot,\cdot]_{\mathfrak{H}})$ and $(\mathfrak{H}, \langle\cdot,\cdot\rangle_{\mathfrak{H}})$ dates back at least to [36, § 4]. We briefly review the essential facts needed for this paper.

Let $S$ be a closed symmetric linear relation in a Krein space $(\mathfrak{H}, [\cdot,\cdot]_{\mathfrak{H}})$ with adjoint $S^*$. We assume that the defect of $S$ is finite; that is $\delta = \dim(S^*/S) \in \mathbb{N}$. Then $JS$ is a closed symmetric linear relation with adjoint $(JS)^{\langle * \rangle} = JS^*$ in the Hilbert space $\big(\mathfrak{H}, \langle \cdot, \cdot \rangle_{\mathfrak{H}}\big)$. The equality $\dim\big((JS^*)/(JS)\big) = \dim(S^*/S) = \delta$ establishes a tight connection between the definite and indefinite settings.

It follows from the definition of $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ that $JS$ being symmetric in $\big(\mathfrak{H}, \langle \cdot, \cdot \rangle_{\mathfrak{H}}\big)$ is equivalent to $JS \subset \mathfrak{H}^2$ being a neutral subspace of $\big(\mathfrak{H}^2, \langle\!\langle \cdot, \cdot \rangle\!\rangle\big)$. Moreover, its adjoint $(JS)^{\langle * \rangle} = JS^*$ in $\big(\mathfrak{H}, \langle \cdot, \cdot \rangle_{\mathfrak{H}}\big)$ is the orthogonal complement of $JS$ in $\big(\mathfrak{H}^2, \langle\!\langle \cdot, \cdot \rangle\!\rangle\big)$ and $JS$ is the isotropic part of $JS^*$. Therefore, the factor space $\big((JS^*)/(JS), \langle\!\langle \cdot, \cdot \rangle\!\rangle\big)$ is a finite-dimensional Pontryagin space. The von Neumann equality

$$JS^* = (JS)^{\langle * \rangle} = JS \oplus \big((JS)^{\langle * \rangle} \cap (-\mathrm{i}I_{\mathfrak{H}})\big) \oplus \big((JS)^{\langle * \rangle} \cap (\mathrm{i}I_{\mathfrak{H}})\big)$$

is a fundamental decomposition of the degenerate inner product space $\big(JS^*, \langle\!\langle \cdot, \cdot \rangle\!\rangle\big)$. The space $\big((JS)^{\langle * \rangle} \cap (\pm\mathrm{i}I_{\mathfrak{H}}), \pm\langle\!\langle \cdot, \cdot \rangle\!\rangle\big)$ is a Hilbert space. The von Neumann equality implies that the Pontryagin spaces

$$\Big((JS^*)/(JS), \langle\!\langle \cdot, \cdot \rangle\!\rangle\Big) \quad \text{and} \quad \Big(\big((JS)^{\langle * \rangle} \cap (-\mathrm{i}I_{\mathfrak{H}})\big) \oplus \big((JS)^{\langle * \rangle} \cap (\mathrm{i}I_{\mathfrak{H}})\big), \langle\!\langle \cdot, \cdot \rangle\!\rangle\Big)$$

are isomorphic. Consequently, these spaces have negative index $d^-$ and positive index $d^+$, where $\dim\big((JS)^{\langle * \rangle} \cap (\pm\mathrm{i}I_{\mathfrak{H}})\big) = \dim \operatorname{nul}\big((JS)^{\langle * \rangle} \mp (\mathrm{i}I_{\mathfrak{H}})\big) = d^\pm$. Clearly, $\delta = d^- + d^+$. The numbers $d^-$ and $d^+$ are called the *defect numbers* of $JS$.

Returning to $S$, we observe that the definition of $[\![ \cdot, \cdot ]\!]$ implies that $S$ being symmetric in $\big(\mathfrak{H}, [\cdot, \cdot]_{\mathfrak{H}}\big)$ is equivalent to $S \subset \mathfrak{H}^2$ being a neutral subspace of $\big(\mathfrak{H}^2, [\![ \cdot, \cdot ]\!]\big)$. Moreover, its adjoint $S^*$ in $\big(\mathfrak{H}, [\cdot, \cdot]_{\mathfrak{H}}\big)$ coincides with the orthogonal complement of $S$ in $\big(\mathfrak{H}^2, [\![ \cdot, \cdot ]\!]\big)$, and $S$ is the isotropic part of $S^*$.

It is straightforward to verify that the mapping $\{f, g\} \mapsto \{f, Jg\}$ defines an isomorphism between the degenerate inner product spaces $\big(S^*, [\![ \cdot, \cdot ]\!]\big)$ and $\big(JS^*, \langle\!\langle \cdot, \cdot \rangle\!\rangle\big)$. Consequently, the quotient spaces $\big(S^*/S, [\![ \cdot, \cdot ]\!]\big)$ and $\big((JS^*)/(JS), \langle\!\langle \cdot, \cdot \rangle\!\rangle\big)$ are isomorphic, implying that $\big(S^*/S, [\![ \cdot, \cdot ]\!]\big)$ is a Pontryagin space with negative index $d^-$, positive index $d^+$, and $\delta = d^- + d^+$. The numbers $d^-$ and $d^+$ are called the *defect numbers* of $S$.

A mapping $\mathsf{b} : S^* \to \mathbb{C}^\delta$ is called a *boundary mapping for* $S$ if it is linear, surjective and $\operatorname{nul} \mathsf{b} = S$. The next lemma concerns the abstract Green's or Lagrange identity for the indefinite inner product $[\![ \cdot, \cdot ]\!]$:

$$\mathrm{i}\big([f, k]_{\mathfrak{H}} - [g, h]_{\mathfrak{H}}\big) = \mathsf{b}(\{h, k\})^* \mathsf{Q} \mathsf{b}(\{f, g\}) \quad \text{for all} \quad \{f, g\}, \{h, k\} \in S^*, \tag{2.2}$$

in which $\mathsf{b}$ is a boundary mapping for $S$ and $\mathsf{Q}$ is a $\delta \times \delta$ matrix called the *Gram matrix associated with* $\mathsf{b}$.

The definition of a boundary mapping and its connection with the Lagrange identity goes back to [7, Definition 1.1].

**Lemma 2.1** *Let $S$ be a closed symmetric linear relation in a Krein space $\big(\mathfrak{H}, [\cdot, \cdot]_{\mathfrak{H}}\big)$ with defect numbers $d^\pm$ and $\delta = d^- + d^+ \in \mathbb{N}$.*

*(i) For each boundary mapping $\mathsf{b}$ of $S$ there exists a unique $\delta \times \delta$ matrix $\mathsf{Q}$ such that (2.2) holds. It is invertible and self-adjoint and it has $d^-$ negative and $d^+$ positive eigenvalues.*

*(ii)* *Let* $\mathsf{b}$ *be a boundary mapping for* $S$ *with Gram matrix* $\mathsf{Q}$*. A function* $\mathsf{b}_1 : S^* \to \mathbb{C}^\delta$ *is a boundary mapping for* $S$ *if and only if there exists an invertible* $\delta \times \delta$ *matrix* $\mathsf{V}$ *such that* $\mathsf{b}_1 = \mathsf{V}^{-1}\mathsf{b}$*. Moreover,* $\mathsf{V}$ *is unique. If* $\mathsf{b}_1 = \mathsf{V}^{-1}\mathsf{b}$*, then its Gram matrix is* $\mathsf{V}^*\mathsf{Q}\mathsf{V}$*. In particular, two boundary mappings* $\mathsf{b}$ *and* $\mathsf{b}_1 = \mathsf{V}^{-1}\mathsf{b}$ *have the same Gram matrix if and only if* $\mathsf{V}^*\mathsf{Q}\mathsf{V} = \mathsf{Q}$*.*

*(iii)* *For each invertible, self-adjoint* $\delta \times \delta$ *matrix* $\mathsf{Q}_1$ *with* $d^-$ *negative and* $d^+$ *positive eigenvalues there exists a boundary mapping* $\mathsf{b}_1$ *for* $S$ *such that* $\mathsf{Q}_1$ *is a Gram matrix for* $\mathsf{b}_1$*.*

In what follows, $\mathsf{Q}$-unitary matrices $\mathsf{V}$, as at the end of part (ii) of the lemma, will play an important role; see, for example, Theorem 2.9 and (4.11), where $\mathsf{V} = \Lambda^{-1}\Gamma$.

***Proof of Lemma 2.1*** (i) Let $\mathsf{b} : S^* \to \mathbb{C}^\delta$ be a boundary mapping for $S$. The mapping induced by $\mathsf{b}$ on the quotient space $S^*/S$ is a linear bijection between the finite-dimensional vector spaces $S^*/S$ and $\mathbb{C}^\delta$; it will also be denoted by $\mathsf{b} : S^*/S \to \mathbb{C}^\delta$.

Let $i, j \in \{1, \dots, \delta\}$, let $e_j$ be the $j$-th column of $\mathsf{I}_\delta$ and let $\mathsf{Q}$ be the $\delta \times \delta$ matrix whose entry in the $i$-th row and $j$-th column is $[\![\mathsf{b}^{-1}(e_j), \mathsf{b}^{-1}(e_i)]\!]$, where $[\![\cdot,\cdot]\!]$ is defined in (2.1). Then (2.2) holds. Since $[\![\cdot,\cdot]\!]$ is a non-degenerate Hermitian form on $S^*/S$, the matrix $\mathsf{Q}$ is invertible and self-adjoint. By defining the weighted dot product $y^*\mathsf{Q}x$ on $\mathbb{C}^\delta \times \mathbb{C}^\delta$, we endow $\mathbb{C}^\delta$ with the structure of a Pontryagin space. Furthermore, $\mathsf{b} : S^*/S \to \mathbb{C}^\delta$ becomes an isomorphism between Pontryagin spaces. Consequently, $\mathsf{Q}$ has $d^-$ negative and $d^+$ positive eigenvalues.

(ii) The "only if" part: Let $\mathsf{b}_1 : S^* \to \mathbb{C}^\delta$ be a boundary mapping for $S$. Then both $\mathsf{b}$ and $\mathsf{b}_1$ can be considered as linear bijections from $S^*/S$ to $\mathbb{C}^\delta$. Then $\mathsf{b}\mathsf{b}_1^{-1}$ is a linear bijection on $\mathbb{C}^\delta$. Let $V$ be the matrix of $\mathsf{b}\mathsf{b}_1^{-1}$ relative to the standard basis of $\mathbb{C}^\delta$. Then the asserted formulas follow. All remaining claims are straightforward.

(iii) Let $\mathsf{Q}_1$ be an arbitrary invertible self-adjoint matrix with $d^-$ negative and $d^+$ positive eigenvalues. By the complex version of Sylvester's law of inertia [34, Theorem 6.11] $\mathsf{Q}_1$ is conjunctive to $\mathsf{Q}$. That is, there exists an invertible $\delta \times \delta$ matrix $\mathsf{V}$ such that $\mathsf{Q} = \mathsf{V}^*\mathsf{Q}_1\mathsf{V}$. By (ii) the mapping $\mathsf{b}_1 = \mathsf{V}\mathsf{b}$ is a boundary mapping for $S$ whose Gram matrix is $\mathsf{Q}_1$. □

***Remark 2.2*** Let $S$ be as in Lemma 2.1 and assume $d = d^- = d^+$. Let $\Gamma_0$ and $\Gamma_1$ be linear operators from $S^*$ to $\mathbb{C}^d$. The triple $\big(\mathbb{C}^d, \Gamma_0, \Gamma_1\big)$ is called a *boundary value space of* $S^*$ or, in more recent publications, a *boundary triplet of* $S^*$ if

$$\mathsf{b} = \begin{bmatrix}\Gamma_0\\ \Gamma_1\end{bmatrix} : S^* \to \begin{matrix}\mathbb{C}^d\\ \oplus\\ \mathbb{C}^d\end{matrix} \quad \text{and} \quad \mathsf{Q} = \begin{bmatrix} 0 & -\mathrm{i}\mathsf{I}_d\\ \mathrm{i}\mathsf{I}_d & 0\end{bmatrix} \tag{2.3}$$

are a boundary mapping for $S$ with corresponding Gram matrix, see [13, 14], [21, Ch. 8] and [5, Ch. 2], where further references can be found. Boundary triplets give a direct access to $\gamma$-fields and Weyl functions associated with $S$, see [20]. Throughout this paper we could choose and fix $\mathsf{b}$ and $\mathsf{Q}$ to be of the form (2.3) or of the form where $\mathsf{Q} = \operatorname{diag}\big(\mathsf{I}_d, -\mathsf{I}_d\big)$. A definite choice would not simplify the presentation and therefore we follow the general formulation as in, for example, [24], [12] and [2].

The following theorem describes the *canonical* self-adjoint extensions of a symmetric linear relation in a Krein space. These are extensions that act in the same space as the symmetric linear relation. The Hilbert space version is well known and goes back at least to [26, Theorem XII.4.30]. We have not been able to find more recent references.

**Theorem 2.3** *Assume $S$ is a closed symmetric linear relation in a Krein space $(\mathfrak{H}, [\cdot,\cdot]_{\mathfrak{H}})$ with adjoint $S^*$ and defect numbers $d^-, d^+ \in \{0\} \cup \mathbb{N}$.*

*(i) $S$ has a canonical self-adjoint extension if and only if $d^- = d^+$, and $S$ is self-adjoint if and only if $d^- = d^+ = 0$.*
*(ii) Assume $d = d^- = d^+ \in \mathbb{N}$ and let $\mathsf{b} : S^* \to \mathbb{C}^{2d}$ be a boundary mapping with Gram matrix $\mathsf{Q}$. The formula*

$$A = \big\{\{f, g\} \in S^* \,:\, \mathsf{Pb}(\{f, g\}) = 0\big\}$$

*gives a bijective correspondence between the canonical self-adjoint extensions $A$ of $S$ and the constant $d \times 2d$ matrices $\mathsf{P}$ (up to multiplication on the left by invertible $d \times d$ matrices) satisfying*

$$\mathsf{PQ}^{-1}\mathsf{P}^* = 0 \quad \textit{and} \quad \operatorname{rank} \mathsf{P} = d.$$

***Proof*** We use the notation introduced above. Recall that $\big(S^*/S, [\![\cdot,\cdot]\!]\big)$ is a $\delta$-dimensional Pontryagin space with negative index $d^-$, positive index $d^+$, and $\delta = d^- + d^+$. Let $A$ be a closed linear relation in $\mathfrak{H}$ such that $S \subseteq A \subseteq S^*$. By definition of the inner product $[\![\cdot,\cdot]\!]$, $A$ is self-adjoint if and only if $A/S$ equals its orthogonal complement in $\big(S^*/S, [\![\cdot,\cdot]\!]\big)$. Since a subspace of a Pontryagin space equals its orthogonal complement if and only if the negative and positive indices of the Pontryagin space are equal, (i) is proved.

Since $\mathsf{b} : S^*/S \to \mathbb{C}^\delta$ is an isomorphism between $\big(S^*/S, [\![\cdot,\cdot]\!]\big)$ and the Pontryagin space $\mathbb{C}^\delta$ with the weighted dot product $y^*\mathsf{Q}x$, the relation $A$ is self-adjoint if and only if $\mathsf{b}(A)$ equals its orthogonal complement in $\mathbb{C}^\delta$. Furthermore, the subspace $\mathsf{b}(A)$ of $\mathbb{C}^\delta$ equals its orthogonal complement if and only if $\mathsf{b}(A) = \operatorname{ran} \mathsf{M}$, where $\mathsf{M}$ is a $2d \times d$ matrix of rank $d$ satisfying $\mathsf{M}^*\mathsf{QM} = 0$ and $\operatorname{ran} \mathsf{M}$ is its range.

Setting $\mathsf{P} = \mathsf{M}^*\mathsf{Q}$, we have $d = \operatorname{rank} \mathsf{P}$, and the condition $\mathsf{M}^*\mathsf{QM} = 0$ implies $\operatorname{ran} \mathsf{M} = \operatorname{nul} \mathsf{P}$ and $\mathsf{PQ}^{-1}\mathsf{P}^* = 0$. Since the correspondence between $A$ and $\mathsf{M}$ is bijective (modulo multiplication on the right by invertible $d \times d$ matrices), (ii) is proved. □

## 2.2 The Polynomial $\mathcal{P}(z)$

For $d \in \mathbb{N}$ and a $d$-tuple $\mu = (\mu_1, \ldots, \mu_d) \in \mathbb{N}^d$ with $\mu_1 \geq \cdots \geq \mu_d \geq 1$ we define, as in [9], the *canonical subspace* $\mathfrak{C}_\mu$ *of* $\mathbb{C}^d[z]$ by

$$\mathfrak{C}_\mu = \Big\{\big[p_1(z) \cdots p_d(z)\big]^\top \in \mathbb{C}^d[z] \,:\, \deg p_j(z) < \mu_j,\ j \in \{1, \ldots, d\}\Big\}$$

and denote its elements by $f_\mu, g_\mu, 0_\mu$, etc. By $S_\mu$ we denote the operator in $\mathfrak{C}_\mu$ of multiplication by $z$. Its graph is given by

$$S_\mu = \Big\{\{f_\mu, g_\mu\} \in \mathfrak{C}_\mu^2 \,:\, zf_\mu(z) - g_\mu(z) \equiv 0\Big\}.$$

The facts that $\mathfrak{C}_\mu$ is finite-dimensional and $S_\mu$ is an operator without eigenvalues play a key role in this paper, see Subsection 4.1.

Below we recall [11, Theorem 2.1]. It shows that a canonical subspace $\mathfrak{C}_\mu$ arises as the reproducing kernel space related to a certain matrix polynomial and, together with $S_\mu$, serves as a model for any finite-dimensional Pontryagin space with a symmetric operator that has no eigenvalues, see Theorem 4.2. Furthermore, this kind of matrix polynomials appears in the characterization of the Shtraus family of extensions of a symmetric linear relation in a Krein space, see Theorem 4.4.

**Theorem 2.4** *Let $d \in \mathbb{N}$. Let $\mathsf{Q}$ be a self-adjoint $2d \times 2d$ matrix with $d$ positive and $d$ negative eigenvalues. Let $\mathcal{P}(z)$ be a $d \times 2d$ matrix polynomial, and for $j \in \{1, \dots, d\}$, let $\mu_j$ denote the degree of the $j$-th row of $\mathcal{P}(z)$. Assume that the pair $\big(\mathsf{Q}, \mathcal{P}(z)\big)$ has the following properties:*

*(a)* $\mathcal{P}(z)\mathsf{Q}^{-1}\mathcal{P}(z^*)^* = 0$ *for all* $z \in \mathbb{C}$.
*(b)* $\operatorname{rank} \mathcal{P}(z) = d$ *for all* $z \in \mathbb{C}$.
*(c)* $\operatorname{rank} \mathsf{P}_\infty = d$.
*(d)* $\mu_1 \geq \cdots \geq \mu_d \geq 1$.

*Then*

*(i) The reproducing kernel Pontryagin space $\mathfrak{K}_{\mathsf{Q},\mathcal{P}}$ with reproducing kernel defined for $z, w \in \mathbb{C}$ by*

$$K_{\mathsf{Q},\mathcal{P}}(z, w) = \begin{cases} \dfrac{\mathrm{i}}{z - w^*}\mathcal{P}(z)\mathsf{Q}^{-1}\mathcal{P}(w)^* & \textit{for} \quad w \neq z^*, \\ \mathrm{i}\,\mathcal{P}'(z)\mathsf{Q}^{-1}\mathcal{P}(z^*)^* & \textit{for} \quad w = z^*, \end{cases} \tag{2.4}$$

*is the canonical subspace $\mathfrak{C}_\mu$ of $\mathbb{C}^d[z]$.*

*(ii) The operator $S_\mu$ is symmetric in the Pontryagin space $\mathfrak{K}_{\mathsf{Q},\mathcal{P}}$, its defect numbers are equal to $d$ and its adjoint is given by*

$$S_\mu^* = \Big\{\{f_\mu, g_\mu\} \in \mathfrak{C}_\mu^2 \,:\, zf_\mu(z) - g_\mu(z) \equiv \mathcal{P}(z)c \textit{ for some } c \in \mathbb{C}^{2d}\Big\}.$$

*(iii) The linear relation*

$$\Big\{\big\{\{f_\mu, g_\mu\}, c\big\} \in \mathfrak{C}_\mu^2 \times \mathbb{C}^{2d} \,:\, zf_\mu(z) - g_\mu(z) \equiv \mathcal{P}(z)c\Big\}$$

*is (the graph of) an operator $\mathsf{b}_{\mu,\mathcal{P}} : S_\mu^* \to \mathbb{C}^{2d}$ which is a boundary mapping for $S_\mu$ with Gram matrix $-\mathsf{Q}$.*

With our extension of $\mathcal{P}(z)$ to $\overline{\mathbb{C}}$ by setting $\mathcal{P}(\infty) = \mathsf{P}_\infty$, conditions (b) and (c) of Theorem 2.4 can be combined into the single requirement rank $\mathcal{P}(z) = d$ for all $z \in \overline{\mathbb{C}}$.

The inner product on $\mathfrak{C}_\mu$, with respect to which $S_\mu$ is symmetric, is determined by the kernel (2.4). Therefore, we will often denote the reproducing kernel space $\mathfrak{K}_{\mathsf{Q},\mathcal{P}}$ by the pair $\big(\mathfrak{C}_\mu, K_{\mathsf{Q},\mathcal{P}}\big)$. The antispace of $\big(\mathfrak{C}_\mu, K_{\mathsf{Q},\mathcal{P}}\big)$ is $\big(\mathfrak{C}_\mu, K_{-\mathsf{Q},\mathcal{P}}\big)$; it will often appear in the theorems below. In the antispace the definitions of $S_\mu$, $S_\mu^*$, and $\mathsf{b}_{\mu,\mathcal{P}}$ remain the same as in the space itself. In the space $(\mathfrak{C}_\mu, K_{-\mathsf{Q},\mathcal{P}})$, the Gram matrix of $\mathsf{b}_{\mu,\mathcal{P}}$ is $\mathsf{Q}$.

Note that in [11, Theorem 2.1(iii)] there is a mistake: the Gram matrix $\mathsf{Q}$ there should be replaced by $-\mathsf{Q}$, as in Theorem 2.4(iii).

***Remark 2.5*** Let $f_\mu, g_\mu \in \mathfrak{C}_\mu$ and $c \in \mathbb{C}^{2d}$. The following two equivalences follow from Theorem 2.4(ii)(iii):

$$(z-\lambda) f_\mu(z) \equiv \mathcal{P}(z)c \quad \Leftrightarrow \quad f_\mu \in \operatorname{nul}\big(S_\mu^* - \lambda\big) \text{ and } c = \mathsf{b}_{\mu,\mathcal{P}}(\{f_\mu, \lambda f_\mu\}),$$
$$-g_\mu(z) \equiv \mathcal{P}(z)c \quad \Leftrightarrow \quad g_\mu \in \operatorname{mul}\big(S_\mu^*\big) \text{ and } c = \mathsf{b}_{\mu,\mathcal{P}}(\{0_\mu, g_\mu\}).$$

In system theory, a matrix polynomial satisfying property (b) in Theorem 2.4 is called *irreducible*, while one satisfying property (c) is termed *row-reduced*; see, for example, [29, p. 378, Section 6.3.2] and [40, Section 2.7], where irreducible column-reduced polynomials play a prominent role. In coding theory, different terminology is used: *basic* for (b), *reduced* for (c), and *canonical* when both properties hold; see [35, Subsection 13.4.3], and also [28, p. 56], where the term *minimal-basic* is used when both properties are satisfied. An analogous concept to (b) appears in control theory under the term *hyper-regular*; see [30, Section 6.5.3].

By [11, Theorem 3.2] the polynomial $\mathcal{P}(z)$ in Theorem 2.4 also has the trivial common null space property:

$$\bigcap_{z \in \mathbb{C}} \operatorname{nul} \mathcal{P}(z) = \{0\}. \tag{2.5}$$

Equivalently, the columns of $\mathcal{P}(z)$ as vector polynomials over $\mathbb{C}$ are linearly independent.

The following lemma concerns statements about nul $\mathcal{P}(z)$ for each $z \in \overline{\mathbb{C}}$ separately.

**Lemma 2.6** *Let $\mathcal{P}(z)$, $\mu$, $\mathfrak{C}_\mu$, $S_\mu$, $S_\mu^*$, and $\mathsf{b}_{\mu,\mathcal{P}}$ be as in Theorem 2.4, let $c \in \mathbb{C}^{2d}$ and $\lambda \in \mathbb{C}$. The following equivalences hold.*

*(i)* $\mathcal{P}(\lambda)c = 0 \quad \Leftrightarrow \quad \mathcal{P}(z)c \equiv (z-\lambda) f_\mu(z)$ *for some* $f_\mu \in \mathfrak{C}_\mu$.
*(ii)* $\mathsf{P}_\infty c = 0 \quad \Leftrightarrow \quad \mathcal{P}(z)c \in \mathfrak{C}_\mu$.

*Moreover,*

$$\operatorname{nul} \mathcal{P}(\lambda) = \mathsf{b}_{\mu,\mathcal{P}}\big(S_\mu^* \cap \lambda I_{\mathfrak{C}_\mu}\big) \quad \textit{for all} \quad \lambda \in \overline{\mathbb{C}},$$

*where* $\infty I_{\mathfrak{C}_\mu} = \{0_\mu\} \times \mathfrak{C}_\mu$.

***Proof*** (i) The implication $\Leftarrow$ is straightforward. To prove the converse, assume $\mathcal{P}(\lambda)c = 0$ and define

$$f_\mu(z) = \begin{cases} \dfrac{1}{z-\lambda}\big(\mathcal{P}(z) - \mathcal{P}(\lambda)\big)c, & z \in \mathbb{C}\backslash\{\lambda\}, \\ \mathcal{P}'(\lambda)c, & z = \lambda. \end{cases}$$

Then $f_\mu \in \mathfrak{C}_\mu$ and $\mathcal{P}(z)c \equiv (z-\lambda) f_\mu(z)$.

(ii) Set $\mathcal{P}_\infty(z) = \operatorname{diag}\big(z^{\mu_1}, \ldots, z^{\mu_d}\big)\mathsf{P}_\infty$ and $\mathcal{R}(z) = \mathcal{P}(z) - \mathcal{P}_\infty(z)$. Then for every $j \in \{1, \ldots, d\}$ we have that the degree of the $j$-th row of $\mathcal{R}(z)$ is $< \mu_j$. Therefore, $\mathcal{R}(z)c \in \mathfrak{C}_\mu$. It follows that $\mathcal{P}(z)c \in \mathfrak{C}_\mu$ if and only if

$$\mathcal{P}(z)c - \mathcal{R}(z)c = \mathcal{P}_\infty(z)c = \operatorname{diag}\big(z^{\mu_1}, \ldots, z^{\mu_d}\big)\mathsf{P}_\infty c \in \mathfrak{C}_\mu.$$

Since $\operatorname{diag}\big(z^{\mu_1}, \ldots, z^{\mu_d}\big)\mathsf{P}_\infty c \in \mathfrak{C}_\mu$ if and only if $\mathsf{P}_\infty c = 0$, the equivalence in (ii) is proved.

The last statement follows from (i), (ii) and Remark 2.5. □

### 2.3 The Equation $\mathcal{W}(z)\mathcal{P}(z) \equiv \mathcal{P}(z)\mathsf{V}$

It appears as one of the equivalent statements in Theorem 4.5. We study it and give examples.

In the next lemma we characterize matrix polynomials whose null spaces coincide for all $z \in \mathbb{C}$.

**Lemma 2.7** *Let $d \in \mathbb{N}$ and let $\mathcal{P}(z)$ and $\mathcal{Q}(z)$ be $d \times 2d$ matrix polynomials of rank $d$ for all $z \in \mathbb{C}$. Then*

$$\operatorname{nul}\mathcal{P}(z) \ = \operatorname{nul}\mathcal{Q}(z) \quad \textit{for all} \quad z \in \mathbb{C}.$$

*if and only if there exists a unimodular $d \times d$ matrix polynomial $\mathcal{W}(z)$ such that*

$$\mathcal{W}(z)\mathcal{P}(z) \equiv \mathcal{Q}(z).$$

***Proof*** The proof of the "if" part is trivial. We continue with the proof of the "only if" part. Since $\mathcal{P}(z)$ has full rank for all $z \in \mathbb{C}$, by the polynomial Smith Normal Form Theorem [29, Section 6.3.3] there exist a unimodular $d \times d$ matrix polynomial $\mathcal{U}(z)$ and a unimodular $2d \times 2d$ matrix polynomial $\mathcal{V}(z)$ such that

$$\mathcal{P}(z) \equiv \mathcal{U}(z)\big[\mathsf{I}_d \ \ \mathsf{0}\big]\mathcal{V}(z) \equiv \big[\mathcal{U}(z) \ \ \mathsf{0}\big]\mathcal{V}(z). \tag{2.6}$$

The factorization in (2.6) yields that the null space of $\mathcal{P}(z)$ is the range of the $2d \times d$ matrix $\mathcal{V}(z)^{-1}\big[\mathsf{0} \ \ \mathsf{I}_d\big]^\top$. Since $\operatorname{nul}\mathcal{P}(z) = \operatorname{nul}\mathcal{Q}(z)$, it follows that

$$\mathcal{Q}(z)\mathcal{V}(z)^{-1}\begin{bmatrix}\mathsf{0}\\ \mathsf{I}_d\end{bmatrix} = \mathsf{0}. \tag{2.7}$$

Using (2.7) we show that the $d \times d$ matrix polynomial $\mathcal{W}(z)$ defined by

$$\mathcal{W}(z) \equiv \mathcal{Q}(z)\mathcal{V}(z)^{-1}\begin{bmatrix}\mathcal{U}(z)^{-1}\\ 0\end{bmatrix}$$

has the asserted property:

$$\begin{aligned}\mathcal{W}(z)\mathcal{P}(z) &\equiv \mathcal{Q}(z)\mathcal{V}(z)^{-1}\begin{bmatrix}\mathcal{U}(z)^{-1}\\ 0\end{bmatrix}\begin{bmatrix}\mathcal{U}(z) & 0\end{bmatrix}\mathcal{V}(z)\\ &\equiv \mathcal{Q}(z)\mathcal{V}(z)^{-1}\begin{bmatrix}\mathsf{I}_d & 0\\ 0 & 0\end{bmatrix}\mathcal{V}(z)\\ &\equiv \mathcal{Q}(z)\mathcal{V}(z)^{-1}\begin{bmatrix}\mathsf{I}_d & 0\\ 0 & \mathsf{I}_d\end{bmatrix}\mathcal{V}(z)\\ &\equiv \mathcal{Q}(z).\end{aligned}$$

Since $\mathcal{Q}(z)$ has full rank for all $z \in \mathbb{C}$, so does the $d \times d$ matrix polynomial $\mathcal{W}(z)$. It follows that the determinant of $\mathcal{W}(z)$ is a polynomial without zeros, yielding that $\mathcal{W}(z)$ is unimodular. □

**Lemma 2.8** *Let $\mathcal{P}(z)$ be a $d \times 2d$ matrix polynomial satisfying (b)–(d) of Theorem 2.4. Let $\mathcal{W}(z)$ be a $d \times d$ matrix function and let $\mathsf{V}$ be an invertible $2d \times 2d$ matrix such that*

$$\mathcal{W}(z)\mathcal{P}(z) \equiv \mathcal{P}(z)\mathsf{V}. \tag{2.8}$$

*Setting $\mathcal{T}(z) \equiv \mathcal{W}(z)\mathcal{P}(z)$ we have:*

- *(i)* $\mathsf{T}_\infty = \mathsf{P}_\infty\mathsf{V}$, $\operatorname{rank}\mathsf{T}_\infty = d$, *and for $j \in \{1,\ldots,d\}$ the degree of the $j$-th row of $\mathcal{T}(z)$ is $\mu_j$.*
- *(ii) The matrix function $\mathcal{W}(z)$ is a unimodular $d \times d$ matrix polynomial and the degree of the scalar polynomial $w_{jk}(z)$ in the $j$-th row and the $k$-th column of $\mathcal{W}(z)$ satisfies the inequality*

$$\deg\bigl(w_{jk}(z)\bigr) \le \mu_j - \mu_k \quad \textit{for all} \quad j,k \in \{1,\ldots,d\}. \tag{2.9}$$

- *(iii) There exists an invertible $d \times d$ matrix $\mathsf{W}$ such that $\mathsf{W}\mathsf{P}_\infty = \mathsf{T}_\infty$.*

***Proof*** (i) The statements follow from $\operatorname{rank}(\mathsf{P}_\infty\mathsf{V}) = d$ and the existence of the limit

$$\lim_{z\to\infty} \operatorname{diag}\bigl(z^{-\mu_1},\ldots,z^{-\mu_d}\bigr)\mathcal{T}(z) = \mathsf{P}_\infty\mathsf{V},$$

see (1.2).

(ii) Since $\mathsf{V}$ is invertible, for every $z \in \mathbb{C}$ the matrix $\mathcal{W}(z)\mathcal{P}(z)$ in (2.8) has rank $d$. Therefore, for every $z \in \mathbb{C}$ the matrix $\mathcal{W}(z)$ is invertible. The identity in (2.8) implies nul $\mathcal{P}(z) = \text{nul}\big(\mathcal{P}(z)\mathsf{V}\big)$ for all $z \in \mathbb{C}$. By Lemma 2.7 there exists a unimodular matrix polynomial $\mathcal{W}_1(z)$ such that $\mathcal{W}_1(z)\mathcal{P}(z) \equiv \mathcal{P}(z)\mathsf{V}$. Now (2.8) and the fact that $\mathcal{P}(z)$ has full rank for all $z \in \mathbb{C}$ imply that $\mathcal{W}(z) \equiv \mathcal{W}_1(z)$ is a unimodular matrix polynomial.

Since $\mathcal{P}(z)$ has properties (b) and (c) of Theorem 2.4, by [31, Theorem A.2] it has the predictable degree property: For every $u(z) = \big[u_1(z) \cdots u_d(z)\big] \in \mathbb{C}^{1\times d}[z]$ we have

$$\deg\big(u(z)\mathcal{P}(z)\big) = \max\big\{\mu_k + \deg\big(u_k(z)\big) : k \in \{1, \ldots, d\}\big\}.$$

Applying this property to each row of $\mathcal{T}(z)$, for every $j \in \{1, \ldots, d\}$ we obtain

$$\deg\big(\mathcal{T}(z)|_j\big) = \max\big\{\mu_k + \deg\big(w_{jk}(z)\big) : k \in \{1, \ldots, d\}\big\},$$

where $\mathcal{T}(z)|_j$ denotes the $j$-th row of $\mathcal{T}(z)$. Since by (i) this maximum equals $\mu_j$, (2.9) follows.

(iii) Consider in $\mathbb{C}^d$ the linear relation $\mathsf{W} = \big\{\{\mathsf{P}_\infty x, \mathsf{T}_\infty x\} : x \in \mathbb{C}^{2d}\big\}$. Since rank $\mathsf{P}_\infty = d$, the dom $\mathsf{W} = \mathbb{C}^d$. From the equivalence between the equalities in the chain of equivalences

| | | | |
|---|---|---|---|
| $\mathsf{P}_\infty x = 0$ | $\Leftrightarrow$ | $\mathcal{P}(z)x \in \mathfrak{C}_\mu$ | by Lemma 2.6 |
| | $\Leftrightarrow$ | $\mathcal{W}(z)\mathcal{P}(z)x \in \mathfrak{C}_\mu$ | by item (ii)) |
| | $\Leftrightarrow$ | $\mathcal{T}(z)x \in \mathfrak{C}_\mu$ | by definition of $\mathcal{T}(z)$ |
| | $\Leftrightarrow$ | $\mathsf{T}_\infty x = 0$ | by Lemma 2.6 |

it follows that $\mathsf{W}$ is the graph of a bijection on $\mathbb{C}^d$, hence $\mathsf{T}_\infty = \mathsf{W}\mathsf{P}_\infty$. □

The next theorem is an addendum to [10, Theorem 6.2]. There we considered a linear bijection on $\mathfrak{C}_\mu$ that intertwines $S_\mu$ and now we assume that it also intertwines $S_\mu^*$.

**Theorem 2.9** *Let* $\mathsf{Q}$, $\mathcal{P}(z)$, $\mu$, $\mathfrak{C}_\mu$, $S_\mu$, $K_{\mathsf{Q},\mathcal{P}}$, *and* $S_\mu^*$ *be as in Theorem* 2.4.

*(i) Let* $W : \mathfrak{C}_\mu \to \mathfrak{C}_\mu$ *be a linear bijection. Then*

$$W S_\mu = S_\mu W \quad \text{and} \quad W S_\mu^* = S_\mu^* W \tag{2.10}$$

*if and only if there exists a unimodular* $d \times d$ *matrix polynomial* $\mathcal{W}(z)$ *and an invertible* $2d \times 2d$ *matrix* $\mathsf{V}$ *satisfying*

$$\mathcal{W}(z)\mathcal{P}(z) \equiv \mathcal{P}(z)\mathsf{V} \tag{2.11}$$

*such that*

$$(W f_\mu)(z) \equiv \mathcal{W}(z) f_\mu(z) \quad \text{for all} \quad f_\mu \in \mathfrak{C}_\mu. \tag{2.12}$$

*(ii) Assume (2.11) and (2.12). Then*

$$W : (\mathfrak{C}_\mu, K_{\mathsf{Q},\mathcal{P}}) \to (\mathfrak{C}_\mu, K_{\mathsf{Q},\mathcal{P}})$$

*is an isomorphism if and only if* $\mathsf{Q} = \mathsf{V}^*\mathsf{Q}\mathsf{V}$.

***Proof*** (i) Assume (2.10). By [10, Theorem 6.2] and the first equality in (2.10) there exists a unimodular $d \times d$ matrix polynomial $\mathcal{W}(z)$ such that (2.12) holds. The algebra of linear relations yields that the equalities in (2.10) are equivalent to: For all $f_\mu, g_\mu \in \mathfrak{C}_\mu$ we have the equivalences

$$\{f_\mu, g_\mu\} \in S_\mu \quad \Leftrightarrow \quad \{Wf_\mu, Wg_\mu\} \in S_\mu$$

and

$$\{f_\mu, g_\mu\} \in S_\mu^* \quad \Leftrightarrow \quad \{Wf_\mu, Wg_\mu\} \in S_\mu^*. \tag{2.13}$$

They show that, like $\mathsf{b}_{\mu,\mathcal{P}}$, also $\mathsf{b}_{\mu,\mathcal{P}}(\{W\cdot, W\cdot\})$ is a boundary mapping for $S_\mu$. Hence Lemma 2.1(ii) implies the existence of an invertible $2d \times 2d$ matrix $\mathsf{V}$ such that

$$\mathsf{b}_{\mu,\mathcal{P}}(\{Wf_\mu, Wg_\mu\}) = \mathsf{V}\mathsf{b}_{\mu,\mathcal{P}}(\{f_\mu, g_\mu\}) \quad \text{for all} \quad \{f_\mu, g_\mu\} \in S_\mu^*. \tag{2.14}$$

By Theorem 2.4(ii),(iii) and (2.12) the equivalence in (2.13) can be restated as

$$zf_\mu(z) - g_\mu(z) \equiv \mathcal{P}(z)\mathsf{b}_{\mu,\mathcal{P}}(\{f_\mu, g_\mu\})$$
$$\Leftrightarrow \quad z\mathcal{W}(z)f_\mu(z) - \mathcal{W}(z)g_\mu(z) \equiv \mathcal{P}(z)\mathsf{b}_{\mu,\mathcal{P}}(\{Wf_\mu, Wg_\mu\}).$$

Multiplying the left-hand side by $\mathcal{W}(z)$ and using (2.14) we obtain

$$\mathcal{W}(z)\mathcal{P}(z)\mathsf{b}_{\mu,\mathcal{P}}(\{f_\mu, g_\mu\}) \equiv \mathcal{P}(z)\mathsf{V}\mathsf{b}_{\mu,\mathcal{P}}(\{f_\mu, g_\mu\}) \quad \text{for all} \quad \{f_\mu, g_\mu\} \in S_\mu^*.$$

Since $\mathsf{b}_{\mu,\mathcal{P}} : S_\mu^* \to \mathbb{C}^{2d}$ is a surjection, this implies (2.11).

To prove the converse, assume (2.11) and (2.12). Then by Lemma 2.8, (2.9) holds. By [10, Theorem 6.2] (2.9) yields that the first equality in (2.10) holds as well. To prove the second equality, we use the following implications

$$\begin{aligned}
\{f_\mu, g_\mu\} \in S_\mu^* &\overset{1}{\Rightarrow} zf_\mu(z) - g_\mu(z) \equiv \mathcal{P}(z)\mathsf{b}_{\mu,\mathcal{P}}(\{f_\mu, g_\mu\}) \\
&\overset{2}{\Rightarrow} z\mathcal{W}(z)f_\mu(z) - \mathcal{W}(z)g_\mu(z) \equiv \mathcal{P}(z)\mathsf{V}\mathsf{b}_{\mu,\mathcal{P}}(\{f_\mu, g_\mu\}) \\
&\overset{3}{\Rightarrow} \{Wf_\mu, Wg_\mu\} \in S_\mu^*.
\end{aligned}$$

The implication $\overset{1}{\Rightarrow}$ follows from Theorem 2.4(ii) and (iii), $\overset{2}{\Rightarrow}$ follows from (2.11) and $\overset{3}{\Rightarrow}$ follows from Theorem 2.4(ii). These implications are in fact equivalences since

$W : \mathfrak{C}_\mu \to \mathfrak{C}_\mu$ is a linear bijection. This proves (2.13) and hence the equivalent second equality in (2.10).

(ii) To prove the last statement of the theorem, assume (2.11), (2.12), and set $\mathcal{T}(z) = \mathcal{W}(z)\mathcal{P}(z)$. By Lemma 2.8(iii) and [10, Theorem 10.4] the operator $W$ is an isomorphism between $\mathfrak{C}_\mu = (\mathfrak{C}_\mu, K_{\mathsf{Q},\mathcal{P}})$ and $\mathfrak{C}_\mu = (\mathfrak{C}_\mu, K_{\mathsf{Q},\mathcal{T}})$.

Assume $\mathsf{Q} = \mathsf{V}^*\mathsf{Q}\mathsf{V}$. Since $\mathsf{Q} = \mathsf{V}^*\mathsf{Q}\mathsf{V}$ is equivalent to $\mathsf{Q}^{-1} = \mathsf{V}\mathsf{Q}^{-1}\mathsf{V}^*$, (2.11) implies that $K_{\mathsf{Q},\mathcal{T}}(z, w) \equiv K_{\mathsf{Q},\mathcal{P}}(z, w)$. Hence, the reproducing kernel spaces $\mathfrak{K}_{\mathsf{Q},\mathcal{P}}$ and $\mathfrak{K}_{\mathsf{Q},\mathcal{T}}$ coincide, and thus $W$ is an isomorphism on $\mathfrak{C}_\mu = \mathfrak{K}_{\mathsf{Q},\mathcal{P}} = (\mathfrak{C}_\mu, K_{\mathsf{Q},\mathcal{P}})$.

To prove the converse, assume that $W$ is an isomorphism on $\mathfrak{C}_\mu = \mathfrak{K}_{\mathsf{Q},\mathcal{P}}$. Since $W$ is also an isomorphism between $\mathfrak{C}_\mu = \mathfrak{K}_{\mathsf{Q},\mathcal{P}}$ and $\mathfrak{C}_\mu = \mathfrak{K}_{\mathsf{Q},\mathcal{T}}$, the two reproducing kernel inner products on $\mathfrak{C}_\mu$ coincide. Hence the adjoints of $S_\mu$ in $\mathfrak{K}_{\mathsf{Q},\mathcal{P}}$ and $\mathfrak{K}_{\mathsf{Q},\mathcal{T}}$ coincide. It follows from Theorem 2.4(ii)(iii) and (2.11) that

$$\mathcal{P}(z)\mathsf{b}_{\mu,\mathcal{P}} = \mathcal{T}(z)\mathsf{b}_{\mu,\mathcal{T}} = \mathcal{P}(z)\mathsf{V}\mathsf{b}_{\mu,\mathcal{T}} \quad \text{for all} \quad z \in \mathbb{C}.$$

Formula (2.5) now implies that $\mathsf{b}_{\mu,\mathcal{T}} = \mathsf{V}^{-1}\mathsf{b}_{\mu,\mathcal{P}}$. Since $-\mathsf{Q}$ is the Gram matrix for $\mathsf{b}_{\mu,\mathcal{P}}$ as well as for $\mathsf{b}_{\mu,\mathcal{T}}$, by Lemma 2.1(iii) there exists an invertible $2d \times 2d$ matrix $\mathsf{U}$ such that $\mathsf{b}_{\mu,\mathcal{T}} = \mathsf{U}^{-1}\mathsf{b}_{\mu,\mathcal{P}}$ and $\mathsf{Q} = \mathsf{U}^*\mathsf{Q}\mathsf{U}$. It follows that $\mathsf{U} = \mathsf{V}$ and $\mathsf{Q} = \mathsf{V}^*\mathsf{Q}\mathsf{V}$. □

### 2.4 Examples

In all examples, $\mathsf{Q}$ is a self-adjoint $2d \times 2d$ matrix with $d$ positive and $d$ negative eigenvalues and $\mathcal{P}(z)$ is a $d \times 2d$ matrix polynomial such that the pair $\big(\mathsf{Q}, \mathcal{P}(z)\big)$ has the properties (a)–(d) of Theorem 2.4. Recall that the degree of $\mathcal{P}(z)$ is $p = \mu_1 \geq 1$.

In each example below we study the solution of the equation

$$\mathcal{W}(z)\mathcal{P}(z) \equiv \mathcal{P}(z)\mathsf{V}, \tag{2.15}$$

where $\mathcal{P}(z)$ is known, and the unknowns are a unimodular $d \times d$ matrix polynomial $\mathcal{W}(z)$ and an invertible $2d \times 2d$ matrix $\mathsf{V}$. It is evident that for any $\alpha \in \mathbb{C}\backslash\{0\}$, the pair $\mathcal{W}(z) \equiv \alpha \mathsf{I}_d$ and $\mathsf{V} = \alpha \mathsf{I}_{2d}$ forms a solution, which we call trivial. In the next example and Example 2.12, we present cases where $\mathcal{P}(z)$ admits only the trivial solution.

***Example 2.10*** We determine the solution of (2.15) in the case $d = 1$. Then $\mathcal{W}(z)$ is a scalar polynomial which we denote by $w(z)$. By Lemma 2.8(ii) $w(z)$ is unimodular and hence equal to a nonzero constant which we denote by $a$. The equation (2.15) becomes $\mathcal{P}(z)\,(\mathsf{V} - a\mathsf{I}_2) \equiv 0$. Now (2.5) implies $\mathsf{V} = a\mathsf{I}_2$. Therefore, for every pair $\big(\mathsf{Q}, \mathcal{P}(z)\big)$ satisfying the properties (a)–(d) of Theorem 2.4 with $d = 1$ the equation (2.15) has only the trivial solution.

Next, we reformulate (2.15) as a matrix equation. If $\mathcal{W}(z)$ satisfies (2.15), then by Lemma 2.8(ii) $\deg \mathcal{W}(z) \leq p - 1$. Consequently, $\mathcal{W}(z)$ can be expressed as

$$\mathcal{W}(z) = \mathsf{W}_0 + \cdots + \mathsf{W}_{p-1}z^{p-1}, \quad \text{where} \quad \mathsf{W}_k \in \mathbb{C}^{d\times d} \quad \text{for all} \quad k \in \{0, \ldots, p-1\}.$$

Define the $2pd \times 2pd$ block Toeplitz matrix $\mathbf{W}$ as follows:

$$\mathbf{W} = \begin{bmatrix} W_0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ W_1 & W_0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ W_{p-1} & W_{p-2} & \cdots & W_0 & 0 & \cdots & 0 & 0 \\ 0 & W_{p-1} & \cdots & W_1 & W_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & W_{p-1} & W_{p-2} & \cdots & W_0 & 0 \\ 0 & 0 & \cdots & 0 & W_{p-1} & \cdots & W_1 & W_0 \end{bmatrix}.$$

Formally, for $j, k \in \{1, \dots, 2p\}$ the $d \times d$ block in $j$-th block-row and $k$-th block-column of $\mathbf{W}$ is $W_{j-k}$ if $j - k \in \{0, \dots, p-1\}$ and $0$ otherwise.

The identity in (2.15) can now be expressed in matrix form as

$$\mathbf{W}C_{\mathcal{P}} = C_{\mathcal{P}}V, \tag{2.16}$$

where $C_{\mathcal{P}}$ is the $2pd \times 2d$ coefficient matrix of $\mathcal{P}(z)$, with the coefficients of powers of $z$ greater than $p$ being the zero matrices.

From (2.5) and (1.1), it follows that the matrix $C_{\mathcal{P}}$ has linearly independent columns. Therefore, the matrix $I_{2pd} - C_{\mathcal{P}}\big(C_{\mathcal{P}}^* C_{\mathcal{P}}\big)^{-1} C_{\mathcal{P}}^*$ is the orthogonal projection onto the orthogonal complement of $\operatorname{ran}(C_{\mathcal{P}})$, the range of $C_{\mathcal{P}}$, in the Euclidean space $\mathbb{C}^{2pd}$. Applying this orthogonal projection to both sides of (2.16) results in the equivalent matrix equation for $\mathbf{W}$:

$$\Big(I_{2pd} - C_{\mathcal{P}}\big(C_{\mathcal{P}}^* C_{\mathcal{P}}\big)^{-1} C_{\mathcal{P}}^*\Big)\mathbf{W}C_{\mathcal{P}} = 0_{2pd \times 2d}.$$

Solving the preceding matrix equation for the $pd^2$ unknown entries of $\mathbf{W}$, we determine that the matrix $V$, which satisfies (2.16), is given by

$$V = \big(C_{\mathcal{P}}^* C_{\mathcal{P}}\big)^{-1} C_{\mathcal{P}}^* \mathbf{W} C_{\mathcal{P}}.$$

If the leading coefficient of $\mathcal{P}(z)$ has full rank, that is $\mu = (p, \dots, p)$, Lemma 2.8(ii) implies that $\mathcal{W}(z)$ is a constant invertible $d \times d$ matrix $W_0$. In this case (2.16) simplifies to

$$\big(I_{p+1} \otimes W_0\big)C_{\mathcal{P}} = C_{\mathcal{P}}V,$$

where $C_{\mathcal{P}}$ is the $(p+1)d \times 2d$ coefficient matrix of $\mathcal{P}(z)$.

***Example 2.11*** Let $\mathcal{P}(z)$ be a $d \times 2d$ matrix binomial of degree $p$ and assume that the pair $\big(\mathsf{Q}, \mathcal{P}(z)\big)$ satisfies conditions (a)–(d) of Theorem 2.4. These conditions are equivalent to the following four conditions:

$$\mathcal{P}(z) = \mathsf{P}_0 + \mathsf{P}_p z^p,$$

the $2d \times 2d$ matrix $\begin{bmatrix} \mathsf{P}_0 \\ \mathsf{P}_p \end{bmatrix}$ is invertible, $\mathsf{P}_0\mathsf{Q}^{-1}\mathsf{P}_0^* = \mathsf{P}_p\mathsf{Q}^{-1}\mathsf{P}_p^* = 0$, and $\mathrm{i}\mathsf{P}_p\mathsf{Q}^{-1}\mathsf{P}_0^*$ is an invertible self-adjoint $d \times d$ matrix.

Since in this case $\mu = (p, \ldots, p)$, if $\mathcal{W}(z)$ satisfies (2.15), then Lemma 2.8(ii) implies $\mathcal{W}(z) \equiv \mathsf{W}_0$, that is, it is a constant invertible matrix. Then (2.15) takes the form

$$\mathsf{W}_0\mathcal{P}(z) \equiv \mathcal{P}(z)\mathsf{V}, \tag{2.17}$$

or, equivalently,

$$\begin{bmatrix} \mathsf{W}_0 & 0 \\ 0 & \mathsf{W}_0 \end{bmatrix} \begin{bmatrix} \mathsf{P}_0 \\ \mathsf{P}_p \end{bmatrix} = \begin{bmatrix} \mathsf{P}_0 \\ \mathsf{P}_p \end{bmatrix} \mathsf{V}.$$

Since $\begin{bmatrix} \mathsf{P}_0 \\ \mathsf{P}_p \end{bmatrix}$ is invertible, the solution for $\mathsf{W}_0$ and $\mathsf{V}$ of (2.17), and thus of (2.15), is

$$\mathsf{V} = \begin{bmatrix} \mathsf{P}_0 \\ \mathsf{P}_p \end{bmatrix}^{-1} \begin{bmatrix} \mathsf{W}_0 & 0 \\ 0 & \mathsf{W}_0 \end{bmatrix} \begin{bmatrix} \mathsf{P}_0 \\ \mathsf{P}_p \end{bmatrix} \quad \text{for all invertible} \quad \mathsf{W}_0 \in \mathbb{C}^{d \times d}.$$

Block matrix calculations show that

$$\mathsf{V}^*\mathsf{Q}\mathsf{V} = \mathsf{Q} \quad \Leftrightarrow \quad \mathsf{W}_0\big(\mathrm{i}\mathsf{P}_p\mathsf{Q}^{-1}\mathsf{P}_0^*\big)\mathsf{W}_0^* = \mathrm{i}\mathsf{P}_p\mathsf{Q}^{-1}\mathsf{P}_0^*.$$

The kernel associated with $\mathcal{P}(z)$ is

$$K_{\mathsf{Q},\mathcal{P}}(z, w) = \mathrm{i}\mathsf{P}_p\mathsf{Q}^{-1}\mathsf{P}_0^* \sum_{k=0}^{p-1} z^{p-1-k} w^{*k}, \quad z, w \in \mathbb{C}.$$

The corresponding reproducing kernel Pontryagin space is the $pd$-dimensional space $\mathbb{C}^d[z]_{<p}$. Its negative and positive index equal to $pd/2$, if $p$ is even. If $p$ is odd, the negative index is $(p-1)d/2 + e_-$ and the positive index is $(p-1)pd/2 + e_+$, where $e_-$ denotes the number of negative and $e_+$ denotes the number of positive eigenvalues of the self-adjoint $d \times d$ matrix $\mathrm{i}\mathsf{P}_p\mathsf{Q}^{-1}\mathsf{P}_0^*$.

***Example 2.12*** Consider the $2 \times 4$ matrix polynomial

$$\mathcal{P}(z) = \begin{bmatrix} 0 & 2z & z^2 & 1 \\ 1 & z^2 & 0 & z \end{bmatrix} \quad \text{with} \quad \mathsf{Q} = \begin{bmatrix} 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{bmatrix}.$$

This pair satisfies conditions (a)–(d) of Theorem 2.4.

Since in this case $\mathsf{P}_2 = \mathsf{P}_\infty$, Lemma 2.8(ii) implies that $\mathcal{W}(z)$ is a constant invertible $2 \times 2$ matrix. That is, $\mathcal{W}(z) \equiv \mathsf{W}_0$. To find $\mathsf{W}_0$ we need to solve

$$\big(\mathsf{I}_6 - \mathsf{C}_{\mathcal{P}}(\mathsf{C}_{\mathcal{P}}^*\mathsf{C}_{\mathcal{P}})^{-1}\mathsf{C}_{\mathcal{P}}^*\big)(\mathsf{I}_3 \otimes \mathsf{W}_0)\mathsf{C}_{\mathcal{P}} = 0_{6\times 4}. \tag{2.18}$$

Setting $\mathsf{W}_0 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and calculating (2.18) yields

$$\begin{bmatrix} \frac{1}{2}b & -c & 0 & \frac{1}{2}(a-d) \\ 0 & 0 & 0 & 0 \\ 0 & \frac{2}{5}(a-d) & -\frac{2}{5}c & \frac{1}{5}b \\ -\frac{1}{2}b & c & 0 & \frac{1}{2}(d-a) \\ 0 & 0 & 0 & 0 \\ 0 & \frac{4}{5}(d-a) & \frac{4}{5}c & -\frac{2}{5}b \end{bmatrix} = 0_{6\times 4}.$$

Hence, the only solution of (2.11) is the trivial solution $\mathsf{W}_0 = a\mathsf{I}_2$ and consequently $\mathsf{V} = a\mathsf{I}_4$ with $a \in \mathbb{C}$.

For a matrix $\mathsf{V} = a\mathsf{I}_4$ to satisfy $\mathsf{V}^*\mathsf{Q}\mathsf{V} = \mathsf{Q}$ we need $a = e^{\mathrm{i}s}$ with $s \in \mathbb{R}$. Thus the only solution for (2.11) which satisfies $\mathsf{V}^*\mathsf{Q}\mathsf{V} = \mathsf{Q}$ is the trivial solution

$$\mathsf{V} = e^{\mathrm{i}s}\mathsf{I}_4 \quad \text{and} \quad \mathcal{W}(z) = e^{\mathrm{i}s}\mathsf{I}_2 \quad \text{with} \quad s \in \mathbb{R}.$$

The kernel associated with $\mathcal{P}(z)$ is

$$K_{\mathsf{Q},\mathcal{P}}(z,w) = -\begin{bmatrix} 2 & w^* - z \\ z - w^* & zw^* \end{bmatrix}, \quad z, w \in \mathbb{C}.$$

The corresponding Pontryagin reproducing kernel space is the 4-dimensional space $\mathbb{C}^2[z]_{<2}$ with the negative index 3 and the positive index 1.

***Example 2.13*** Consider the $3 \times 6$ matrix polynomial $\mathcal{P}(z)$ and the $6 \times 6$ matrix $\mathsf{Q}$ as follows:

$$\mathcal{P}(z) = \begin{bmatrix} 1 & 0 & z^3+1 & 0 & -z & -z^2 \\ 2z & z^2 & 2z & 1 & -z^2 & -1 \\ 0 & 1 & 0 & z & 0 & -z \end{bmatrix}, \quad \mathsf{Q} = \begin{bmatrix} 0 & \mathrm{i} & 0 & 0 & -\mathrm{i} & 0 \\ -\mathrm{i} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{i} \\ \mathrm{i} & 0 & \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} & 0 & 0 \end{bmatrix}.$$

This pair satisfies conditions (a)–(d) of Theorem 2.4 and we studied it in [10, Example 12.2].

Wolfram Mathematica gives the following solutions for (2.15):

$$\mathsf{V} = \begin{bmatrix} c & 0 & 0 & -b & a & b \\ 0 & c & 0 & 0 & 0 & 0 \\ 0 & a & c & b & -a & -b \\ -2a & -b & -2a & c & 0 & 0 \\ 0 & 0 & 0 & -2a & c & 2a \\ -2a & -b & -2a & 0 & 0 & c \end{bmatrix}, \quad \mathcal{W}(z) = \begin{bmatrix} c & az & a+bz^2 \\ 0 & c & 2az \\ 0 & 0 & c \end{bmatrix},$$

where $a, b, c \in \mathbb{C}$ with $c \neq 0$. Substituting $\mathsf{V}$ into the equation $\mathsf{V}^*\mathsf{Q}\mathsf{V} = \mathsf{Q}$ and solving, we obtain a $\mathsf{Q}$-unitary $\mathsf{V}$,

$$\mathsf{V} = e^{\mathrm{i}s} \begin{bmatrix} 1 & 0 & 0 & \alpha^2 - \mathrm{i}\beta & \mathrm{i}\alpha & -\alpha^2 + \mathrm{i}\beta \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \mathrm{i}\alpha & 1 & -\alpha^2 + \mathrm{i}\beta & -\mathrm{i}\alpha & \alpha^2 - \mathrm{i}\beta \\ -2\mathrm{i}\alpha & \alpha^2 - \mathrm{i}\beta & -2\mathrm{i}\alpha & 1 & 0 & 0 \\ 0 & 0 & 0 & -2\mathrm{i}\alpha & 1 & 2\mathrm{i}\alpha \\ -2\mathrm{i}\alpha & \alpha^2 - \mathrm{i}\beta & -2\mathrm{i}\alpha & 0 & 0 & 1 \end{bmatrix},$$

and the corresponding $\mathcal{W}(z)$,

$$\mathcal{W}(z) = e^{\mathrm{i}s} \begin{bmatrix} 1 & \mathrm{i}\alpha z & \mathrm{i}\alpha + (-\alpha^2 + \mathrm{i}\beta) z^2 \\ 0 & 1 & 2\mathrm{i}\alpha z \\ 0 & 0 & 1 \end{bmatrix},$$

where $s, \alpha, \beta \in \mathbb{R}$.

In this example $d = 3$ and $\mu = (3, 2, 1)$. The kernel associated with $\mathcal{P}(z)$ is

$$K_{\mathsf{Q},\mathcal{P}}(z, w) = -\begin{bmatrix} 1 - zw^*(w^* + z) & w^* - z & z^2 \\ z - w^* & 2zw^* & 0 \\ (w^*)^2 & 0 & 0 \end{bmatrix}, \quad z, w \in \mathbb{C}.$$

The corresponding Pontryagin reproducing kernel space is the 6-dimensional space $\big(\mathbb{C}[z]_{<3}\big) \oplus \big(\mathbb{C}[z]_{<2}\big) \oplus \mathbb{C}$ with the negative index 2 and the positive index 4.

## 3 The Coupling Theorem and Shtraus Subspaces

### 3.1 The Coupling Theorem

The *Coupling Theorem* formulated below is a copy of [12, Theorem 5.1]. A similar theorem for boundary triplets appeared in [18, Theorem 6.4], [19, Theorem 5.3] and is subsequently generalized in [33]. It is formulated in a Krein space setting. We leave to the reader the straightforward reduction to the original Hilbert space version, applying fundamental symmetries as in Subsection 2.1.

**Theorem 3.1** *For $k \in \{1, 2\}$ let $S_k$ be a closed symmetric linear relation in a Krein space $(\mathfrak{H}_k, [\,\cdot\,,\cdot\,]_{\mathfrak{H}_k})$ with defect numbers $d_k^-, d_k^+ \in \{0\} \cup \mathbb{N}$, $\delta_k = d_k^- + d_k^+$.*

*(i) $S_1 \oplus S_2$ has a canonical self-adjoint extension $\widetilde{A}$ in the direct sum Krein space $\widetilde{\mathfrak{H}} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ such that $\widetilde{A} \cap \mathfrak{H}_k^2 = S_k$ for all $k \in \{1, 2\}$ if and only if*

$$d_1^+ = d_2^- \quad \text{and} \quad d_1^- = d_2^+. \tag{3.1}$$

*(ii) Assume that (3.1) holds. If $S_1$ ($S_2$) is self-adjoint in $\mathfrak{H}_1$ ($\mathfrak{H}_2$), then $S_2$ ($S_1$) is self-adjoint, $\widetilde{A}$ in (i) is unique and given by $\widetilde{A} = S_1 \oplus S_2$.*

*(iii) Assume that (3.1) holds and that $S_1$ and $S_2$ are not self-adjoint. Set $\delta = \delta_1 = \delta_2 \in \mathbb{N}$. Let $\mathsf{b}_k : S_k^* \to \mathbb{C}^\delta$ be a boundary mapping for $S_k$ with Gram matrix $\mathsf{Q}_k$, $k \in \{1, 2\}$. The formula*

$$\widetilde{A} = \left\{ \left\{ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \right\} : \begin{array}{l} \{f_1, g_1\} \in S_1^*, \{f_2, g_2\} \in S_2^*, \\ \mathsf{b}_1(\{f_1, g_1\}) + \Gamma \mathsf{b}_2(\{f_2, g_2\}) = 0 \end{array} \right\} \tag{3.2}$$

*gives a bijection between all canonical self-adjoint extensions $\widetilde{A}$ of $S_1 \oplus S_2$ in $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ with $\widetilde{A} \cap \mathfrak{H}_k^2 = S_k$, $k \in \{1, 2\}$, and all $\delta \times \delta$ invertible matrices $\Gamma$ with*

$$\mathsf{Q}_2 + \Gamma^* \mathsf{Q}_1 \Gamma = 0. \tag{3.3}$$

Note that we do not assume that the resolvent set $\rho(\widetilde{A})$ of $\widetilde{A}$ is nonempty. The operator $\widetilde{A}$ defined by (3.2) with $\Gamma$ satisfying (3.3) will often be denoted by $\widetilde{A}_\Gamma$.

The defining relation in (3.2) and formula (3.3) can be written as

$$\begin{bmatrix} \mathsf{I}_\delta & \Gamma \end{bmatrix} \begin{bmatrix} \mathsf{b}_1(\{f_1, g_1\}) \\ \mathsf{b}_2(\{f_2, g_2\}) \end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix} \mathsf{I}_\delta & \Gamma \end{bmatrix} \begin{bmatrix} \mathsf{Q}_1^{-1} & 0 \\ 0 & \mathsf{Q}_2^{-1} \end{bmatrix} \begin{bmatrix} \mathsf{I}_\delta \\ \Gamma^* \end{bmatrix} = 0.$$

According to Theorem 2.3, these equalities show that $\widetilde{A}$ is indeed a canonical self-adjoint extension of $S_1 \oplus S_2$ in $\widetilde{\mathcal{H}} = \mathcal{H}_1 \oplus \mathcal{H}_2$.

### 3.2 The Family of Shtraus Subspaces

The self-adjoint relation $\widetilde{A}$ in the Coupling Theorem can also be viewed as an extension in the direct sum space $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ of the symmetric linear relation $S_1$ in the smaller space $\mathfrak{H}_1$ and then $\mathfrak{H}_2$ is called the *exit space of* $\widetilde{A}$. It is often with this in mind that one associates with $\widetilde{A}$ the following family of linear relations $T_{\widetilde{A}}(z)$, $z \in \overline{\mathbb{C}}$:

$$T_{\widetilde{A}}(z) := \begin{cases} \left\{ \{f_1, g_1\} : \left\{ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \right\} \in \widetilde{A},\ g_2 - zf_2 = 0 \right\}, & z \in \mathbb{C}, \\ \left\{ \{f_1, g_1\} : \left\{ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \right\} \in \widetilde{A},\ f_2 = 0 \right\}, & z = \infty. \end{cases} \tag{3.4}$$

In [23] it is named after A.V. Shtraus [37, 38] and called the *family of Shtraus subspaces associated with* $\widetilde{A}$. It is studied in for example [22], [23], [18, 19] and [25]. Each member of the family is a kind of compression of $\widetilde{A}$ to $\mathfrak{H}_1$ and satisfies $S_1 \subset T_{\widetilde{A}}(z) \subset S_1^*$ for all $z \in \overline{\mathbb{C}}$. For this reason the family is also called the *family of Shtraus extensions of* $S_1$ *associated with* $\widetilde{A}$. The inclusions and the fact that $\dim S_1^*/S_1 = 2d$ with $d \in \mathbb{N}$ imply that all members of the family are closed. For $z \in \mathbb{C}$ we have the resolvent relations

$$\left(T_{\widetilde{A}}(z) - zI_{\mathfrak{H}_1}\right)^{-1} = \left\{ \{g_1 - zf_1, f_1\} : \left\{ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \right\} \in \widetilde{A},\ g_2 - zf_2 = 0 \right\},$$

$$\left(\widetilde{A} - zI_{\widetilde{\mathfrak{H}}}\right)^{-1} = \left\{ \left\{ \begin{bmatrix} g_1 - zf_1 \\ g_2 - zf_2 \end{bmatrix}, \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right\} : \left\{ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \right\} \in \widetilde{A} \right\},$$

and hence the first resolvent coincides with generalized (or compressed) resolvent of the associated self-adjoint relation:

$$\left(T_{\widetilde{A}}(z) - zI_{\mathfrak{H}_1}\right)^{-1} = \widetilde{P}_{\mathfrak{H}_1}\left(\widetilde{A} - zI_{\widetilde{\mathfrak{H}}}\right)^{-1}\big|_{\mathfrak{H}_1}, \quad z \in \mathbb{C},$$

where $\widetilde{P}_{\mathfrak{H}_1}$ is the orthogonal projection in $\widetilde{\mathfrak{H}}$ onto $\mathfrak{H}_1$. The linear relation $T_{\widetilde{A}}(\infty)$ is the compression of $\widetilde{A}$ to $\mathfrak{H}_1$ and often written as

$$C_{\mathfrak{H}_1}(\widetilde{A}) := T_{\widetilde{A}}(\infty) = \widetilde{P}_{\mathfrak{H}_1}\widetilde{A}\,\big|_{\mathrm{dom}(\widetilde{A} \cap \mathfrak{H}_1)}.$$

It can be viewed as the formal limit

$$T_{\widetilde{A}}(\infty) = \lim_{z\to 0} T_{\widetilde{A}}(1/z) = \lim_{z\to 0}\Big\{\big\{\widetilde{P}_{\mathfrak{H}}\widetilde{f},\, \widetilde{P}_{\mathfrak{H}_1}\widetilde{g}\big\} \,:\, \big\{\widetilde{f},\widetilde{g}\big\}\in\widetilde{A},\ z\widetilde{g}-\widetilde{f}\in\mathfrak{H}_1\Big\}.$$

If $\dim\mathfrak{H}_2 \in \mathbb{N}$, then, by the indefinite version [3, Theorem 4.1] of W. Stenger's lemma [39, Lemma 1], the compression $C_{\mathfrak{H}_1}(\widetilde{A})$ is self-adjoint in $\mathfrak{H}_1$. The compression has also been studied, for example, in [23], [32], [1], and [25].

### 3.3 A Polynomial Description of Shtraus Subspaces

When one of the components in the Coupling Theorem is a canonical subspace, the family of Shtraus subspaces admits a polynomial description.

**Theorem 3.2** *Let $d\in\mathbb{N}$ and let $\mathsf{Q}_1$ and $\mathsf{Q}$ be self-adjoint $2d\times 2d$ matrices with $d$ positive and $d$ negative eigenvalues. Let $(\mathfrak{H}_1,[\,\cdot,\cdot\,]_{\mathfrak{H}_1})$ be a Krein space and let $S_1$ be a closed symmetric linear relation with defect numbers equal to $d$ and a boundary mapping $\mathsf{b}_1 : S_1^* \to \mathbb{C}^{2d}$ with Gram matrix $\mathsf{Q}_1$. Let $\mathcal{P}(z)$ be a $d\times 2d$ matrix polynomial such that the pair $\big(\mathsf{Q},\mathcal{P}(z)\big)$ has the properties (a)–(d) of Theorem 2.4 and let $\mu$, $\mathfrak{C}_\mu$, $S_\mu$, $K_{\mathsf{Q},\mathcal{P}}$, $S_\mu^*$, and $\mathsf{b}_{\mu,\mathcal{P}}$ be as in Theorem 2.4.*

*Let $\widetilde{B}_\Gamma$ be the linear relation in the Krein space $\mathfrak{H}\oplus\big(\mathfrak{C}_\mu, K_{\mathsf{Q},\mathcal{P}}\big)$ defined by*

$$\widetilde{B}_\Gamma = \left\{\left\{\begin{bmatrix} f_1\\ f_\mu\end{bmatrix},\begin{bmatrix} g_1\\ g_\mu\end{bmatrix}\right\} \,:\, \begin{array}{l}\{f_1,g_1\}\in S_1^*,\{f_\mu,g_\mu\}\in S_\mu^*,\\ \mathsf{b}_1(\{f_1,g_1\})+\Gamma\mathsf{b}_{\mu,\mathcal{P}}(\{f_\mu,g_\mu\})=0\end{array}\right\}, \tag{3.5}$$

*where $\Gamma$ is an invertible $2d\times 2d$ matrix such that $-\mathsf{Q}+\Gamma^*\mathsf{Q}_1\Gamma=0$.*

*Then $\widetilde{B}_\Gamma$ is self-adjoint in the Krein space $\mathfrak{H}_1\oplus\mathfrak{C}_\mu$, $\widetilde{B}_\Gamma\cap\mathfrak{H}_1^2=S_1$, $\widetilde{B}_\Gamma\cap\mathfrak{C}_\mu^2=S_\mu$ and the family of Shtraus subspaces associated with $\widetilde{B}_\Gamma$ is given by*

$$T_{\widetilde{B}_\Gamma}(\lambda)=\Big\{\{f_1,g_1\}\in S_1^* \,:\, \mathcal{P}(\lambda)\Gamma^{-1}\mathsf{b}_1(\{f_1,g_1\})=0\Big\} \ \text{for all}\ \ \lambda\in\overline{\mathbb{C}}. \tag{3.6}$$

***Proof*** The linear relation $\widetilde{B}_\Gamma$ is self-adjoint by Theorem 3.1(iii). The equalities for $S_1$ and $S_\mu$ follow immediately from (3.5) and the definition of a boundary mapping.

We prove (3.6) for $\lambda\in\mathbb{C}$. According to the definition in (3.4), $\{f_1,g_1\}\in T_{\widetilde{B}_\Gamma}(\lambda)$ if and only if $\{f_1,g_1\}\in S_1^*$ and

$$\exists f_\mu\in\mathfrak{C}_\mu \ \text{ such that }\ \{f_\mu,\lambda f_\mu\}\in S_\mu^* \ \text{ and }\ \mathsf{b}_1(\{f_1,g_1\})+\Gamma\mathsf{b}_{\mu,\mathcal{P}}(\{f_\mu,\lambda f_\mu\})=0.$$

By Theorem 2.4(iii) the latter holds if and only if $\{f_1,g_1\}\in S_1^*$ and

$$\exists f_\mu\in\mathfrak{C}_\mu \ \text{ such that }\ (z-\lambda)f_\mu(z)\equiv\mathcal{P}(z)\mathsf{b}_{\mu,\mathcal{P}}(\{f_\mu,\lambda f_\mu\})\equiv-\mathcal{P}(z)\Gamma^{-1}\mathsf{b}_1(\{f_1,g_1\}).$$

By Lemma 2.6(i) this is equivalent to $\{f_1,g_1\}$ belonging to the right-hand side of (3.6). This proves the equality (3.6).

We now prove (3.6) for $\lambda = \infty$. By (3.4), $\{f_1, g_1\} \in T_{\widetilde{B}_\Gamma}(\infty)$ if and only if $\{f_1, g_1\} \in S_1^*$ and

$$\exists g_\mu \in \mathfrak{C}_\mu \ \text{ such that } \ \{0, g_\mu\} \in S_\mu^* \ \text{ and } \ \mathsf{b}_1(\{f_1, g_1\}) + \Gamma \mathsf{b}_{\mu,\mathcal{P}}(\{0, g_\mu\}) = 0.$$

By Theorem 2.4(iii) the latter holds if and only if $\{f_1, g_1\} \in S_1^*$ and

$$\exists g_\mu \in \mathfrak{C}_\mu \ \text{ such that } \ -g_\mu(z) \equiv \mathcal{P}(z)\mathsf{b}_{\mu,\mathcal{P}}(\{0, g_\mu\}) \equiv -\mathcal{P}(z)\Gamma^{-1}\mathsf{b}_1(\{f_1, g_1\}).$$

Thus, $\{f_1, g_1\} \in T_{\widetilde{B}_\Gamma}(\infty)$ if and only if $\{f_1, g_1\} \in S_1^*$ and $\mathcal{P}(z)\Gamma^{-1}\mathsf{b}_1(\{f_1, g_1\}) \in \mathfrak{C}_\mu$. By Lemma 2.6(ii) this condition is equivalent to the equality in (3.6) for $\lambda = \infty$. □

# 4 Main Results

## 4.1 Main Assumptions

In this paper we study the families of Shtraus subspaces associated with the self-adjoint extensions $\widetilde{A}$ described in the Coupling Theorem 3.1. Thereby we make the following *main assumptions*:

$$\dim \mathfrak{H}_2 \in \mathbb{N} \quad \text{and} \quad \sigma_p(S_2) = \emptyset. \tag{4.1}$$

By $\sigma_p(S_2)$ we denote the point spectrum of $S_2$ in $\overline{\mathbb{C}}$; by definition $\infty \in \sigma_p(S_2)$ if and only if $S_2$ is not an operator.

The assumptions (4.1) are related to the requirement that $\widetilde{A}$ is *minimal with respect to* $\mathfrak{H}_1$, or that $\widetilde{A}$ and $\mathfrak{H}_1$ are *closely connected* (see [23, p. 462]), which means that $\rho(\widetilde{A}) \neq \emptyset$ and

$$\widetilde{\mathfrak{H}} = \overline{\operatorname{span}}\Big\{u + \big(\widetilde{A} - zI_{\widetilde{\mathfrak{H}}}\big)^{-1} v \ : \ u, v \in \mathfrak{H}_1, \ \ z \in \rho\big(\widetilde{A}\big) \cap \mathbb{C}\Big\}.$$

In fact, the following result holds.

**Proposition 4.1** *Let $\widetilde{A}$ be a self-adjoint relation with $\rho(\widetilde{A}) \neq \emptyset$ in $\widetilde{\mathfrak{H}} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, the direct sum of the Krein spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, and set $S_2 = \widetilde{A} \cap \mathfrak{H}_2^2$.*

*(i) If $\widetilde{A}$ is minimal with respect to $\mathfrak{H}_1$, then $\sigma_p(S_2) = \emptyset$.*
*(ii) If (4.1) holds, then $\widetilde{A}$ is minimal with respect to $\mathfrak{H}_1$.*

***Proof*** We use that $\rho(\widetilde{A})$ is symmetric relative to the real axis: for all $z \in \overline{\mathbb{C}}$ we have $z \in \rho(\widetilde{A})$ if and only if $z^* \in \rho(\widetilde{A})$.

(i) Assume $\sigma_p(S_2) \neq \emptyset$ and $\lambda \in \sigma_p(S_2)$. If $\lambda = \infty$, then there exists $f_2 \neq 0$ such that $\{0, f_2\} \in S_2$. Then $\{0, f_2\} \in \widetilde{A}$ and, consequently, $(\widetilde{A} - zI_{\widetilde{\mathfrak{H}}})^{-1} f_2 = 0$ for all $z \in \rho(\widetilde{A}) \cap \mathbb{C}$. Therefore, for all $u, v \in \mathfrak{H}_1$ and all $z \in \rho(\widetilde{A}) \cap \mathbb{C}$ we have

$$\big[u + \big(\widetilde{A} - zI_{\widetilde{\mathfrak{H}}}\big)^{-1} v, f_2\big]_{\widetilde{\mathfrak{H}}} = \big[v, \big(\widetilde{A} - zI_{\widetilde{\mathfrak{H}}}\big)^{-1} f_2\big]_{\widetilde{\mathfrak{H}}} = 0,$$

proving that $\widetilde{A}$ is not minimal.

If $\lambda \in \mathbb{C}$ and $\lambda \in \sigma_p(S_2)$, then $\lambda \notin \rho(\widetilde{A})$ and there exists $f_2 \neq 0$ such that $\{f_2, \lambda f_2\} \in S_2$. Consequently, $\lambda^* \notin \rho(\widetilde{A})$ and $\{f_2, \lambda f_2\} \in \widetilde{A}$. Furthermore, for all $z^* \in \rho(\widetilde{A}) \cap \mathbb{C}$ we have $\big(\widetilde{A} - z^* I_{\widetilde{\mathfrak{H}}}\big)^{-1} f_2 = \frac{1}{\lambda - z^*} f_2$. Therefore for all $u, v \in \mathfrak{H}_1$ and for all $z \in \rho(\widetilde{A}) \cap \mathbb{C}$ we have

$$\big[u + \big(\widetilde{A} - zI_{\widetilde{\mathfrak{H}}}\big)^{-1} v, f_2\big]_{\widetilde{\mathfrak{H}}} = \frac{1}{\lambda^* - z}\big[v, f_2\big]_{\widetilde{\mathfrak{H}}} = 0,$$

proving that $\widetilde{A}$ is not minimal. Thus, we proved that $\sigma_p(S_2) \neq \emptyset$ implies $\widetilde{A}$ is not minimal, which is equivalent to (i).

(ii) Assume that $\dim \mathfrak{H}_2 \in \mathbb{N}$ and that $\widetilde{A}$ is not minimal. Then there exists $\widetilde{g} \in \widetilde{\mathfrak{H}}$ such that $\widetilde{g} \neq 0$ and

$$\big[u + \big(\widetilde{A} - zI_{\widetilde{\mathfrak{H}}}\big)^{-1} v, \widetilde{g}\big]_{\widetilde{\mathfrak{H}}} = 0 \quad \text{for all} \quad u, v \in \mathfrak{H}_1 \quad \text{and for all} \quad z \in \rho\big(\widetilde{A}\big) \cap \mathbb{C}.$$

Then $\big[u, \widetilde{g}\big]_{\widetilde{\mathfrak{H}}} = 0$ for all $u \in \mathfrak{H}_1$, implying that $\widetilde{g} \in \mathfrak{H}_2$, and

$$\big[v, \big(\widetilde{A} - z^* I_{\widetilde{\mathfrak{H}}}\big)^{-1} \widetilde{g}\big]_{\widetilde{\mathfrak{H}}} = 0 \quad \text{for all} \quad v \in \mathfrak{H} \quad \text{and for all} \quad z \in \rho\big(\widetilde{A}\big) \cap \mathbb{C}.$$

Therefore, for all $z \in \rho\big(\widetilde{A}\big) \cap \mathbb{C}$ we have $\big(\widetilde{A} - zI_{\widetilde{\mathfrak{H}}}\big)^{-1} \widetilde{g} \in \mathfrak{H}_2$.

Set

$$\mathfrak{M} = \operatorname{span}\Big(\big\{\widetilde{g}\big\} \cup \Big\{\big(\widetilde{A} - zI_{\widetilde{\mathfrak{H}}}\big)^{-1} \widetilde{g} : z \in \rho\big(\widetilde{A}\big) \cap \mathbb{C}\Big\}\Big).$$

Since the spanning set for $\mathfrak{M}$ is a subset of $\mathfrak{H}_2$, we have that $\mathfrak{M}$ is a nontrivial finite-dimensional closed subspace of $\mathfrak{H}_2$.

Let $w \in \rho\big(\widetilde{A}\big) \cap \mathbb{C}$. Next we will prove that $\mathfrak{M}$ is invariant under the resolvent $\big(\widetilde{A} - wI_{\widetilde{\mathfrak{H}}}\big)^{-1}$. It is sufficient to prove that $\big(\widetilde{A} - wI_{\widetilde{\mathfrak{H}}}\big)^{-1}$ maps the spanning set for $\mathfrak{M}$ into $\mathfrak{M}$. It is clear that $\big(\widetilde{A} - wI_{\widetilde{\mathfrak{H}}}\big)^{-1} \widetilde{g} \in \mathfrak{M}$. Recall the resolvent identity [8, Proposition VI.1.4]:

$$\big(\widetilde{A} - wI_{\widetilde{\mathfrak{H}}}\big)^{-1} - \big(\widetilde{A} - zI_{\widetilde{\mathfrak{H}}}\big)^{-1} = (w - z)\big(\widetilde{A} - wI_{\widetilde{\mathfrak{H}}}\big)^{-1}\big(\widetilde{A} - zI_{\widetilde{\mathfrak{H}}}\big)^{-1},$$

which implies that for $z \in \rho\big(\widetilde{A}\big) \cap \mathbb{C}$ such that $z \neq w$ we have

$$\big(\widetilde{A} - wI_{\widetilde{\mathfrak{H}}}\big)^{-1}\big(\widetilde{A} - zI_{\widetilde{\mathfrak{H}}}\big)^{-1} \widetilde{g} = \frac{1}{w - z}\Big(\big(\widetilde{A} - wI_{\widetilde{\mathfrak{H}}}\big)^{-1} \widetilde{g} - \big(\widetilde{A} - zI_{\widetilde{\mathfrak{H}}}\big)^{-1} \widetilde{g}\Big) \in \mathfrak{M}. \tag{4.2}$$

Taking the limit as $z \to w$ in (4.2), by the continuity of the resolvent, see [8, Theorem VI.1.7], the element of $\mathfrak{M}$ in (4.2) converges to

$$\big(\widetilde{A} - wI_{\widetilde{\mathfrak{H}}}\big)^{-1}\big(\widetilde{A} - wI_{\widetilde{\mathfrak{H}}}\big)^{-1} \widetilde{g} \in \mathfrak{M},$$

which is an element of $\mathfrak{M}$ since $\mathfrak{M}$ is closed. Thus, $(\widetilde{A} - wI_{\widetilde{\mathfrak{H}}})^{-1}\mathfrak{M} \subseteq \mathfrak{M}$.

Since $\dim \mathfrak{M} \in \mathbb{N}$, the restriction of $(\widetilde{A} - wI_{\widetilde{\mathfrak{H}}})^{-1}$ to $\mathfrak{M}$ has an eigenvalue $\mu \in \mathbb{C}$ and a corresponding eigenvector $f_2 \in \mathfrak{M}\backslash\{0\}$; that is $(\widetilde{A} - wI_{\widetilde{\mathfrak{H}}})^{-1} f_2 = \mu f_2$. Thus

$$\{\mu f_2, (w\mu + 1) f_2\} \in \widetilde{A} \cap \mathfrak{H}_2^2 = S_2.$$

Now, either $\infty \in \sigma_p(S_2)$ (if $\mu = 0$) or $(w + 1/\mu) \in \sigma_p(S_2)$ (if $\mu \neq 0$), proving that $\sigma_p(S_2) \neq \emptyset$. In conclusion, we proved that $\dim \mathfrak{H}_2 \in \mathbb{N}$ and $\widetilde{A}$ is not minimal imply $\sigma_p(S_2) \neq \emptyset$. This implication is equivalent to (ii). □

The main assumptions (4.1) rule out the possibility that $\widetilde{A}$ takes the form described in part (ii) of the Coupling Theorem 3.1. Specifically, since in part (ii) the operator $S_2$ is self-adjoint, (4.1) implies that $S_2$, being an operator, is densely defined and hence everywhere defined on $\mathfrak{H}_2$, as $\mathfrak{H}_2$ is finite dimensional. Consequently, $S_2$ must have an eigenvalue, which contradicts the second part of (4.1). Therefore, we shall only consider the case where $\widetilde{A} = \widetilde{A}_\Gamma$, as described in part (iii) of Theorem 3.1.

The main assumptions (4.1) and [10, Lemma 3.4] imply that the defect numbers of $S_2$ are equal and nonzero, hence all four defect numbers of $S_1$ and $S_2$ are equal to the same number $d \in \mathbb{N}$, say, that is,

$$d = d_k^\pm \quad \text{and} \quad \delta = \delta_k = 2d \quad \text{for all} \quad k \in \{1, 2\}. \tag{4.3}$$

Finally, the main assumptions (4.1) also imply the existence of a $d \times 2d$ matrix polynomial $\mathcal{P}(z)$ as in Theorem 2.4 with which we construct a canonical subspace model $(\mathfrak{C}_\mu, S_\mu)$ of the pair $(\mathfrak{H}_2, S_2)$, see Theorem 4.2. The model in turn is used to describe the Shtraus family in terms of $\mathcal{P}(z)$ and the parameter $\Gamma$ from (3.3), see (4.7) in Theorem 4.4.

### 4.2 The Model Theorem

In this section we use the notation of Theorem 2.4. The following result, which we call the *Model Theorem*, states that the quadruple $(\mathfrak{C}_\mu, K_{-\mathsf{Q},\mathcal{P}}, S_\mu, \mathsf{b}_{\mu,\mathcal{P}})$ is a model for $(\mathfrak{G}, [\,\cdot\,,\cdot\,]_{\mathfrak{G}}, S, \mathsf{b})$ under the assumptions analogous to (4.1): $\dim \mathfrak{G} \in \mathbb{N}$ and $\sigma_p(S) = \emptyset$.

**Theorem 4.2** *Let* $(\mathfrak{G}, [\,\cdot\,,\cdot\,]_{\mathfrak{G}})$ *be a finite-dimensional Pontryagin space and let $S$ be a symmetric operator in $\mathfrak{G}$ without eigenvalues. Then the defect numbers of $S$ coincide and are equal to* $d = \operatorname{codim}(\operatorname{dom} S)$. *Let* $\mathsf{b} : S^* \to \mathbb{C}^{2d}$ *be a boundary map for $S$ with Gram matrix* $\mathsf{Q}$. *Moreover, $S$ is nilpotent.*

*Let $m$ denote the nilpotency index of $S$, and define*

$$\delta_j = \dim(\operatorname{dom} S^{j-1}), \quad j \in \{1, \ldots, m+1\}, \quad (\text{so that } \delta_1 = \dim \mathfrak{G},\ \delta_{m+1} = 0).$$

*Let* $\mu = (\mu_1, \mu_2, \ldots, \mu_d)$ *be the $d$-tuple with entries*

$$\mu_j = \#\{i \in \{1, \ldots, m\} : \delta_i - \delta_{i+1} \geq j\}, \quad j \in \{1, \ldots, d\},$$

*so that* $\mu_1 \geq \cdots \geq \mu_d \geq 1$.

*Then there exists a* $d \times 2d$ *matrix polynomial* $\mathcal{P}(z)$ *such that the pair* $\big(-\mathsf{Q}, \mathcal{P}(z)\big)$ *satisfies (a)–(d) of Theorem 2.4 and there exists an isomorphism*

$$\Phi : \big(\mathfrak{G}, [\,\cdot\,,\cdot\,]_{\mathfrak{G}}\big) \to (\mathfrak{C}_\mu, K_{-\mathsf{Q},\mathcal{P}})$$

*such that:*

*(a)* $\Phi S = S_\mu \Phi$, *and*
*(b) for all* $\{f, g\} \in S^*$ *we have* $\{\Phi f, \Phi g\} \in S_\mu^*$ *and* $\mathsf{b}(\{f, g\}) = \mathsf{b}_{\mu,\mathcal{P}}(\{\Phi f, \Phi g\})$.

*The matrix polynomial* $\mathcal{P}(z)$ *is unique up to multiplication on the left by a unimodular* $d \times d$ *matrix polynomial* $\mathcal{W}(z) = \big[w_{jk}(z)\big]_{j,k=1}^{d}$ *satisfying (2.9).*

The theorem implies that the positive (negative) index of $\mathfrak{G}$ is equal to the number of positive (negative) squares of the kernel $K_{-\mathsf{Q},\mathcal{P}}$ in (2.4), $\dim \mathfrak{G} = \sum_{j=1}^{d} \mu_j$ and the nilpotency index of $S$ equals $m = \mu_1 = \deg \mathcal{P}$.

***Remark 4.3*** Concerning more general models, in [16, Theorem 3.3] a model is presented of a closed simple entire symmetric operator in a Pontryagin space as the multiplication operator by the independent variable in a de Branges-Pontryagin space determined by an entire de Branges $d \times 2d$ matrix function $\mathcal{P}(z)$ (not necessarily a polynomial) and $\mathsf{Q} = \operatorname{diag}\big(\mathsf{I}_d, -\mathsf{I}_d\big)$. For this case the formula analogous to $S_\mu^*$ in Theorem 2.4(ii) is given in [15, Theorem 4.3] in terms of $\mathcal{P}(z)$. We refer to [17] for isometric analogs of results in [15] and [16].

***Proof of Theorem 4.2*** By [10, Theorem 11.1] there exists a $d \times 2d$ matrix polynomial $\mathcal{T}(z)$ such that the pair $\big(-\mathsf{Q}, \mathcal{T}(z)\big)$ satisfies (a)–(d) of Theorem 2.4. Hence $K_{-\mathsf{Q},\mathcal{T}}$ is well defined, and there exists an isomorphism

$$\Phi : \big(\mathfrak{G}, [\,\cdot\,,\cdot\,]_{\mathfrak{G}}\big) \to \big(\mathfrak{C}_\mu, K_{-\mathsf{Q},\mathcal{T}}\big) \tag{4.4}$$

such that $\Phi S = S_\mu \Phi$. Taking the adjoint of both sides of $\Phi S = S_\mu \Phi$, and using $\Phi^* = \Phi^{-1}$ we obtain $\Phi S^* = S_\mu^* \Phi$. Hence for all $\{f, g\} \in S^*$ we have $\{\Phi f, \Phi g\} \in S_\mu^*$.

Consequently, the mapping defined by

$$\mathsf{b}_1(\{f, g\}) = \mathsf{b}_{\mu,\mathcal{T}}(\{\Phi f, \Phi g\})$$

is defined for all $\{f, g\} \in S^*$, it is linear and surjective with $\operatorname{nul} \mathsf{b}_1 = S$. Hence $\mathsf{b}_1$ is a boundary mapping for $S$. Using the fact that $\Phi$ is an isomorphism and that, by Theorem 2.4(iii), $\mathsf{Q}$ is the Gram matrix of $\mathsf{b}_{\mu,\mathcal{T}}$ we have

$$\begin{aligned}
\mathrm{i}\big([f, k]_{\mathfrak{G}} - [g, h]_{\mathfrak{G}}\big) &= \mathrm{i}\big([\Phi f, \Phi k]_{\mathfrak{C}_\mu} - [\Phi g, \Phi h]_{\mathfrak{C}_\mu}\big) \\
&= \mathsf{b}_{\mu,\mathcal{T}}(\{\Phi h, \Phi k\})^* \mathsf{Q} \mathsf{b}_{\mu,\mathcal{T}}(\{\Phi f, \Phi g\}) \\
&= \mathsf{b}_1(\{h, k\})^* \mathsf{Q} \mathsf{b}_1(\{f, g\})
\end{aligned}$$

for all $\{f, g\}, \{h, k\} \in S^*$, proving that $\mathsf{Q}$ is the Gram matrix of $\mathsf{b}_1$. Thus the boundary mappings $\mathsf{b}$ and $\mathsf{b}_1$ for $S$ have the same Gram matrix $\mathsf{Q}$. By Lemma 2.1(iii) there exists an invertible $2d \times 2d$ matrix $\mathsf{V}$ such that $\mathsf{V}^*\mathsf{Q}\mathsf{V} = \mathsf{Q}$ and

$$\mathsf{b}(\{f, g\}) = \mathsf{V}^{-1}\mathsf{b}_1(\{f, g\}) = \mathsf{V}^{-1}\mathsf{b}_{\mu,\mathcal{T}}(\{\Phi f, \Phi g\}) \quad \text{for all} \quad \{f, g\} \in S^*. \tag{4.5}$$

Define the matrix polynomial $\mathcal{P}(z)$ by $\mathcal{P}(z) \equiv \mathcal{T}(z)\mathsf{V}$. Since $\mathsf{V}$ is invertible, the pair $\big(-\mathsf{Q}, \mathcal{P}(z)\big)$ has properties (a)–(d) of Theorem 2.4. The equality $\mathsf{V}^*\mathsf{Q}\mathsf{V} = \mathsf{Q}$ implies that $K_{-\mathsf{Q},\mathcal{P}}(z, w) \equiv K_{-\mathsf{Q},\mathcal{T}}(z, w)$. Hence $\Phi$ from (4.4) is an isomorphism between the Pontryagin spaces $(\mathfrak{G}, [\,\cdot\,,\cdot\,]_{\mathfrak{G}})$ and $\big(\mathfrak{C}_\mu, K_{-\mathsf{Q},\mathcal{P}}\big)$ and (a)) holds.

From the definition of $\mathcal{P}(z)$ and the defining formula of the boundary mapping for $S_\mu$ in Theorem 2.4(iii) we deduce that for all $\{f, g\} \in S^*$

$$\begin{aligned} z\big(\Phi f\big)(z) - \big(\Phi g\big)(z) &\equiv \mathcal{T}(z)\mathsf{b}_{\mu,\mathcal{T}}(\{\Phi f, \Phi g\}) \\ &\equiv \mathcal{P}(z)\mathsf{V}^{-1}\mathsf{b}_{\mu,\mathcal{T}}(\{\Phi f, \Phi g\}), \end{aligned}$$

hence $\mathsf{V}^{-1}\mathsf{b}_{\mu,\mathcal{T}}(\{\Phi f, \Phi g\}) = \mathsf{b}_{\mu,\mathcal{P}}(\{\Phi f, \Phi g\})$. It follows from this equality and (4.5) that (b)) holds as well.

We prove uniqueness. Let $\mathcal{S}(z)$ be a $d \times 2d$ matrix polynomial such that the pair $\big(-\mathsf{Q}, \mathcal{S}(z)\big)$ has properties (a)–(d) of Theorem 2.4 and let

$$\Psi : (\mathfrak{G}, [\,\cdot\,,\cdot\,]_{\mathfrak{G}}) \to (\mathfrak{C}_\mu, K_{-\mathsf{Q},\mathcal{S}})$$

be an isomorphism for which (a)) and (b)) hold. Set $W = \Psi\Phi^{-1}$. Then

$$W : (\mathfrak{C}_\mu, K_{-\mathsf{Q},\mathcal{P}}) \to (\mathfrak{C}_\mu, K_{-\mathsf{Q},\mathcal{S}})$$

is an isomorphism, and it follows from (a)) that $WS_\mu = S_\mu W$. Moreover, it follows from (b)) that

$$\mathsf{b}_{\mu,\mathcal{P}}(\{f_\mu, g_\mu\}) = \mathsf{b}_{\mu,\mathcal{S}}(\{Wf_\mu, Wg_\mu\}) \quad \text{for all} \quad \{f_\mu, g_\mu\} \in S_\mu^*. \tag{4.6}$$

By [10, Theorem 6.2] there exists a $d \times d$ unimodular matrix $\mathcal{W}(z)$ which satisfies (2.9) and such that $W$ is the operator of multiplication by $\mathcal{W}(z)$. Again from the defining formula of the boundary mapping for $S_\mu$ in Theorem 2.4(iii) and from (4.6) we obtain that for all $\{f_\mu, g_\mu\} \in S_\mu^*$

$$\begin{aligned} \mathcal{W}(z)\mathcal{P}(z)\mathsf{b}_{\mu,\mathcal{P}}(\{f_\mu, g_\mu\}) &\equiv z\mathcal{W}(z)f_\mu(z) - \mathcal{W}(z)g_\mu(z) \\ &\equiv \mathcal{S}(z)\mathsf{b}_{\mu,\mathcal{S}}(\{Wf_\mu, Wg_\mu\}) \\ &\equiv \mathcal{S}(z)\mathsf{b}_{\mu,\mathcal{P}}(\{f_\mu, g_\mu\}). \end{aligned}$$

Since $\mathsf{b}_{\mu,\mathcal{P}} : S_\mu^* \to \mathbb{C}^{2d}$ is a surjection, this implies that $\mathcal{S}(z) \equiv \mathcal{W}(z)\mathcal{P}(z)$. □

### 4.3 Shtraus Subspaces in Terms of $\mathcal{P}(z)$ and $\Gamma$

We now come to the first main result of this paper.

**Theorem 4.4** *Assume that in the setting of the Coupling Theorem 3.1(iii) we have* $\dim \mathfrak{H}_2 \in \mathbb{N}$ *and* $S_2$ *is a symmetric operator without eigenvalues. Then the defect numbers of* $S_1$ *and* $S_2$ *are all equal to* $d$*, so that (4.3) holds:* $d = d_k^- = d_k^+$ *and* $\delta_k = \delta = 2d$ *for* $k \in \{1, 2\}$*. Let* $\mathcal{P}(z)$ *be a* $d \times 2d$ *matrix polynomial such that the pair* $\big(-\mathsf{Q}_2, \mathcal{P}(z)\big)$ *satisfies (a)–(d) of Theorem 2.4 and* $\big(\mathfrak{C}_\mu, K_{-\mathsf{Q}_2,\mathcal{P}}, S_\mu, \mathsf{b}_{\mu,\mathcal{P}}\big)$ *is a model for the quadruple* $\big(\mathfrak{H}_2, [\,\cdot\,,\cdot\,]_{\mathfrak{H}_2}, S_2, \mathsf{b}_2\big)$ *as in Theorem 4.2. Then the family of Shtraus subspaces associated with* $\widetilde{A}_\Gamma$ *defined in (3.2), where* $\Gamma$ *satisfies (3.3), is given by the formula*

$$T_{\widetilde{A}_\Gamma}(\lambda) = \Big\{ \{f_1, g_1\} \in S_1^* \,:\, \mathcal{P}(\lambda)\Gamma^{-1}\mathsf{b}_1(\{f_1, g_1\}) = 0 \Big\} \quad \textit{for all} \quad \lambda \in \overline{\mathbb{C}}. \tag{4.7}$$

Formula (4.7) does not maintain the bijection between the families of Shtraus subspaces defined by the self-adjoint extensions $\widetilde{A}_\Gamma$ in the Coupling Theorem 3.1(iii) on the one hand and the matrices $\Gamma$ in (3.3) on the other hand. The reason is that the family of Shtraus subspaces in (4.7) is characterized as a nulspace involving $\mathcal{P}(\lambda)\Gamma^{-1}$ : $T_{\widetilde{A}_\Gamma}(\lambda) = \mathrm{nul}\big(\mathcal{P}(\lambda)\Gamma^{-1}\mathsf{b}_1\big)$ where $\mathcal{P}(\lambda)\Gamma^{-1}\mathsf{b}_1$ is a mapping from $S_1^*$ to $\mathbb{C}^d$. Finally note that if $T_{\widetilde{A}_\Gamma}(\lambda)$ is also described as the null space

$$T_{\widetilde{A}_\Gamma}(\lambda) = \mathrm{nul}\big(\mathcal{T}(\lambda)\mathsf{b}_1\big), \quad \lambda \in \overline{\mathbb{C}},$$

for some $d \times 2d$ matrix polynomial $\mathcal{T}(z)$, then

$$\mathrm{nul}\big(\mathcal{P}(\lambda)\Gamma^{-1}\big) = \mathsf{b}\big(T_{\widetilde{A}_\Gamma}(\lambda)\big) = \mathrm{nul}\,\mathcal{T}(\lambda), \quad \lambda \in \mathbb{C}.$$

Hence, by Lemma 2.7 there is a unimodular $d \times d$ matrix polynomial $\mathcal{W}(z)$ such that $\mathcal{W}(z)\mathcal{T}(z) \equiv \mathcal{P}(z)\Gamma^{-1}$.

***Proof of Theorem 4.4*** Let $\lambda \in \mathbb{C}$. By the definition in (3.4), $\{f_1, g_1\} \in T_{\widetilde{A}_\Gamma}(\lambda)$ if and only if $\{f_1, g_1\} \in S_1^*$ and

$$\exists f_2 \in \mathfrak{H}_2 \text{ such that } \{f_2, \lambda f_2\} \in S_2^* \text{ and } \mathsf{b}_1(\{f_1, g_1\}) + \Gamma\mathsf{b}_2(\{f_2, \lambda f_2\}) = 0. \tag{4.8}$$

In the proof of Theorem 3.2 we proved that $\{f_1, g_1\} \in T_{\widetilde{B}_\Gamma}(\lambda)$ if and only if $\{f_1, g_1\} \in S_1^*$ and

$$\exists f_\mu \in \mathfrak{C}_\mu \text{ such that } \{f_\mu, \lambda f_\mu\} \in S_\mu^* \text{ and } \mathsf{b}_1(\{f_1, g_1\}) + \Gamma\mathsf{b}_{\mu,\mathcal{P}}(\{f_\mu, \lambda f_\mu\}) = 0. \tag{4.9}$$

According to the Model Theorem 4.2, there exists an isomorphism

$$\Phi : \big(\mathfrak{H}_2, [\,\cdot\,,\cdot\,]_{\mathfrak{H}_2}\big) \to \big(\mathfrak{C}_\mu, K_{-\mathsf{Q}_2,\mathcal{P}}\big)$$

such that $\{f_2, g_2\} \in S_2^*$ if and only if $\{\Phi f_2, \Phi g_2\} \in S_\mu^*$ and furthermore

$$\mathsf{b}_2(\{f_2, g_2\}) = \mathsf{b}_{\mu,\mathcal{P}}(\{\Phi f_2, \Phi g_2\}) \quad \text{for all} \quad \{f_2, g_2\} \in S_2^*.$$

This implies that (4.9) and (4.8) are equivalent. Consequently, $T_{\widetilde{A}_\Gamma}(\lambda) = T_{\widetilde{B}_\Gamma}(\lambda)$ for all $\lambda \in \mathbb{C}$. Similarly, $T_{\widetilde{A}_\Gamma}(\infty) = T_{\widetilde{B}_\Gamma}(\infty)$. With these equalities, the theorem follows from Theorem 3.2. □

## 4.4 Five Equivalent Statements

**Theorem 4.5** *Assume the setting of Theorem 4.4. Let $\Gamma$ and $\Lambda$ be invertible $2d \times 2d$ matrices satisfying $\mathsf{Q}_2 + \Gamma^*\mathsf{Q}_1\Gamma = 0$ and $\mathsf{Q}_2 + \Lambda^*\mathsf{Q}_1\Lambda = 0$. The following statements are equivalent.*

*(i) For all $\lambda \in \mathbb{C}$ we have $T_{\widetilde{A}_\Lambda}(\lambda) = T_{\widetilde{A}_\Gamma}(\lambda)$.*

*(ii) There exists an isomorphism $\Psi : \mathfrak{H}_2 \to \mathfrak{H}_2$ such that $S_2\Psi = \Psi S_2$ and*

$$\big(I_{\mathcal{H}_1} \oplus \Psi\big)\widetilde{A}_\Gamma = \widetilde{A}_\Lambda\big(I_{\mathcal{H}_1} \oplus \Psi\big).$$

*(iii) There exists an isomorphism $W : \big(\mathfrak{C}_\mu, K_{-\mathsf{Q}_2,\mathcal{P}}\big) \to \big(\mathfrak{C}_\mu, K_{-\mathsf{Q}_2,\mathcal{P}}\big)$ such that $WS_\mu = S_\mu W$ and*

$$\big(I_{\mathcal{H}_1} \oplus W\big)\widetilde{B}_\Gamma = \widetilde{B}_\Lambda\big(I_{\mathcal{H}_1} \oplus W\big), \tag{4.10}$$

*where $\widetilde{B}_\Gamma$ and $\widetilde{B}_\Lambda$ are self-adjoint relations introduced in (3.5) in the Krein space $\mathfrak{H}_1 \oplus \big(\mathfrak{C}_\mu, K_{-\mathsf{Q}_2,\mathcal{P}}\big)$.*

*(iv) There exists a $d \times d$ matrix polynomial $\mathcal{W}(z)$ such that*

$$\mathcal{W}(z)\mathcal{P}(z) \equiv \mathcal{P}(z)\mathsf{V}, \tag{4.11}$$

*where $\mathsf{V} = \Lambda^{-1}\Gamma$ satisfies $\mathsf{V}^*\mathsf{Q}_2\mathsf{V} = \mathsf{Q}_2$.*

*(v) There exists a linear bijection $\Theta : \mathfrak{H}_2 \to \mathfrak{H}_2$ such that $\Theta S_2 = S_2\Theta$, $\Theta S_2^* = S_2^*\Theta$ and*

$$\Lambda\mathsf{b}_2(\{\Theta f_2, \Theta g_2\}) = \Gamma\mathsf{b}_2(\{f_2, g_2\}) \quad \textit{for all} \quad \{f_2, g_2\} \in S_2^*.$$

*If any of the above equivalent statements hold, then $T_{\widetilde{A}_\Gamma}(\infty) = T_{\widetilde{A}_\Lambda}(\infty)$.*

***Proof*** Throughout the proof, we use the notation from Theorem 4.4 and its proof. First we introduce an isomorphism that will be used in several items in this proof. Since the quadruple $\big(\mathfrak{C}_\mu, K_{-\mathsf{Q}_2,\mathcal{P}}, S_\mu, \mathsf{b}_{\mu,\mathcal{P}}\big)$ is a model for $\big(\mathfrak{H}_2, [\,\cdot\,,\cdot\,]_{\mathfrak{H}_2}, S_2, \mathsf{b}_2\big)$ as in Theorem 4.2, there exists an isomorphism

$$\Phi : \big(\mathfrak{H}_2, [\,\cdot\,,\cdot\,]_{\mathfrak{H}_2}\big) \to \big(\mathfrak{C}_\mu, K_{-\mathsf{Q}_2,\mathcal{P}}\big) \tag{4.12}$$

such that $\Phi S_2^* = S_\mu^* \Phi$ and $\mathsf{b}_2(\{f_2, g_2\}) = \mathsf{b}_{\mu,\mathcal{P}}(\{\Phi f_2, \Phi g_2\})$ for all $\{f_2, g_2\} \in S_2^*$. Then

$$I_{\mathfrak{H}_1} \oplus \Phi \,:\, \mathfrak{H}_1 \oplus \mathfrak{H}_2 \;\to\; \mathfrak{H}_1 \oplus \big(\mathfrak{C}_\mu, K_{-\mathsf{Q}_2,\mathcal{P}}\big)$$

is an isomorphism of Krein spaces such that for every $\Gamma$ satisfying $\mathsf{Q}_2 + \Gamma^*\mathsf{Q}_1\Gamma = 0$ we have

$$\widetilde{B}_\Gamma\big(I_{\mathfrak{H}_1} \oplus \Phi\big) = \big(I_{\mathfrak{H}_1} \oplus \Phi\big)\widetilde{A}_\Gamma,$$

where $\widetilde{B}_\Gamma$ is defined in (3.5) and $\widetilde{A}_\Gamma$ is defined in (3.2). Since the only boundary mapping for $S_\mu$ in this proof is $\mathsf{b}_{\mu,\mathcal{P}}$, we abbreviate it as $\mathsf{b}_\mu$. By Theorem 2.4 its Gram matrix is $\mathsf{Q}_2$.

(iv)⇒(iii). Assume (iv). Then $\mathcal{W}(z)\mathcal{P}(z) \equiv \mathcal{P}(z)\mathsf{V}$ and $\mathsf{V}^*\mathsf{Q}_2\mathsf{V} = \mathsf{Q}_2$. Since $\mathsf{V}$ is invertible, Lemma 2.8(ii) implies that $\mathcal{W}(z)$ is unimodular. By Theorem 2.9 it follows that the operator

$$W : \big(\mathfrak{C}_\mu, K_{-\mathsf{Q}_2,\mathcal{P}}\big) \to \big(\mathfrak{C}_\mu, K_{-\mathsf{Q}_2,\mathcal{P}}\big) \tag{4.13}$$

of multiplication by $\mathcal{W}(z)$ is an isomorphism such that $WS_\mu = S_\mu W$ and $WS_\mu^* = S_\mu^* W$.

To prove (4.10) we first prove: For all $\{f_\mu, g_\mu\} \in S_\mu^*$ we have

$$\mathsf{V}\mathsf{b}_\mu(\{f_\mu, g_\mu\}) = \mathsf{b}_\mu(\{Wf_\mu, Wg_\mu\}). \tag{4.14}$$

By the definition of the boundary operator $\mathsf{b}_\mu$ in Theorem 2.4(iii) the following identities hold for all $\{f_\mu, g_\mu\} \in S_\mu^*$:

$$\begin{aligned}\mathcal{P}(z)\mathsf{b}_\mu(\{Wf_\mu, Wg_\mu\}) &\equiv z\mathcal{W}(z)f_\mu(z) - \mathcal{W}(z)g_\mu(z)\\ &\equiv \mathcal{W}(z)\mathcal{P}(z)\mathsf{b}_\mu(\{f_\mu, g_\mu\})\\ &\equiv \mathcal{P}(z)\mathsf{V}\mathsf{b}_\mu(\{f_\mu, g_\mu\}),\end{aligned}$$

which, by (2.5), yields (4.14). Now equality (4.10) can be obtained as follows:

$$\begin{aligned}&\big(I_{\mathcal{H}_1} \oplus W\big)\widetilde{B}_\Gamma\big(I_{\mathcal{H}_1} \oplus W^{-1}\big)\\ &\quad= \left\{\left\{\begin{bmatrix} f_1\\ Wf_\mu\end{bmatrix}, \begin{bmatrix} g_1\\ Wg_\mu\end{bmatrix}\right\} : \left\{\begin{bmatrix} f_1\\ f_\mu\end{bmatrix}, \begin{bmatrix} g_1\\ g_\mu\end{bmatrix}\right\} \in \widetilde{B}_\Gamma\right\}\\ &\quad= \left\{\left\{\begin{bmatrix} f_1\\ Wf_\mu\end{bmatrix}, \begin{bmatrix} g_1\\ Wg_\mu\end{bmatrix}\right\} : \begin{array}{l}\{f_1, g_1\} \in S_1^*, \{f_\mu, g_\mu\} \in S_\mu^*,\\ \mathsf{b}_1(\{f_1, g_1\}) + \Gamma\mathsf{b}_\mu(\{f_\mu, g_\mu\}) = 0\end{array}\right\}\\ &\quad= \left\{\left\{\begin{bmatrix} f_1\\ Wf_\mu\end{bmatrix}, \begin{bmatrix} g_1\\ Wg_\mu\end{bmatrix}\right\} : \begin{array}{l}\{f_1, g_1\} \in S_1^*, \{f_\mu, g_\mu\} \in S_\mu^*,\\ \mathsf{b}_1(\{f_1, g_1\}) + \Gamma\mathsf{V}^{-1}\mathsf{b}_\mu(\{Wf_\mu, Wg_\mu\}) = 0\end{array}\right\}\end{aligned}$$

$$= \left\{ \left\{ \begin{bmatrix} f_1 \\ u_\mu \end{bmatrix}, \begin{bmatrix} g_1 \\ v_\mu \end{bmatrix} \right\} : \begin{array}{l} \{f_1, g_1\} \in S_1^*, \{u_\mu, v_\mu\} \in S_\mu^*, \\ \mathsf{b}_1(\{f_1, g_1\}) + \Lambda \mathsf{b}_\mu(\{u_\mu, v_\mu\}) = 0 \end{array} \right\}$$
$$= \widetilde{B}_\Lambda.$$

(iii)⇒(ii). Assume (iii) and define $\Psi : \mathfrak{H}_2 \to \mathfrak{H}_2$ by $\Psi = \Phi^{-1} W \Phi$. Then $\Psi$ is a Krein space isomorphism satisfying $\Psi S_2 = S_2 \Psi$. We have

$$I_{\mathfrak{H}_1} \oplus \Psi = \big(I_{\mathfrak{H}_1} \oplus \Phi^{-1} W\big)\big(I_{\mathfrak{H}_1} \oplus \Phi\big) = \big(I_{\mathfrak{H}_1} \oplus \Phi^{-1}\big)\big(I_{\mathfrak{H}_1} \oplus W\big)\big(I_{\mathfrak{H}_1} \oplus \Phi\big).$$

Consequently,

$$\begin{aligned} \big(I_{\mathfrak{H}_1} \oplus \Psi\big)\widetilde{A}_\Gamma &= \big(I_{\mathfrak{H}_1} \oplus \Phi^{-1} W\big)\widetilde{B}_\Gamma\big(I_{\mathfrak{H}_1} \oplus \Phi\big) \\ &= \big(I_{\mathfrak{H}_1} \oplus \Phi^{-1}\big)\widetilde{B}_\Lambda\big(I_{\mathfrak{H}_1} \oplus W\Phi\big) \\ &= \widetilde{A}_\Lambda\big(I_{\mathfrak{H}_1} \oplus \Psi\big), \end{aligned}$$

which proves (ii).

(ii)⇒(i). First notice that the equality in (ii) is equivalent to the equality

$$\widetilde{A}_\Lambda = \left\{ \left\{ \begin{bmatrix} f_1 \\ \Psi f_2 \end{bmatrix}, \begin{bmatrix} g_1 \\ \Psi g_2 \end{bmatrix} \right\} : \left\{ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \right\} \in \widetilde{A}_\Gamma \right\}. \tag{4.15}$$

Assume (ii). Let $\lambda \in \mathbb{C}$ be arbitrary. By (4.15), $\{f_1, g_1\} \in T_{\widetilde{A}_\Lambda}(\lambda)$ is equivalent to the statement: there exists $\left\{ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \right\} \in \widetilde{A}_\Gamma$ with $\lambda \Psi f_2 - \Psi g_2 = 0$. Since $\Psi$ is an isomorphism, the last statement is equivalent to: there exists $\left\{ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \right\} \in \widetilde{A}_\Gamma$ with $\lambda f_2 - g_2 = 0$, which in turn is equivalent to $\{f_1, g_1\} \in T_{\widetilde{A}_\Gamma}(\lambda)$. Since $\lambda \in \mathbb{C}$ was arbitrary, (i) is proved.

(i)⇒(iv). Assume (i). Then, according to Theorem 4.4,

$$\operatorname{nul}\big(\mathcal{P}(z)\Gamma^{-1}\big) = \mathsf{b}_1\big(T_{\widetilde{A}_\Gamma}(z)\big) = \mathsf{b}_1\big(T_{\widetilde{A}_\Lambda}(z)\big) = \operatorname{nul}\big(\mathcal{P}(z)\Lambda^{-1}\big) \quad \text{for all} \quad z \in \mathbb{C}.$$

By Lemma 2.7 there exists a unimodular $d \times d$ matrix polynomial $\mathcal{W}(z)$ such that (4.11) holds. This proves (iv).

(iv)⇒(v). Assume (iv) and introduce the isomorphism $W$ as in (4.13) and notice its property in (4.14). Recall the property of the isomorphism $\Phi$ in (4.12): for all $\{f_2, g_2\} \in S_2^*$ we have $\mathsf{b}_2(\{f_2, g_2\}) = \mathsf{b}_\mu(\{\Phi f_2, \Phi g_2\})$. Equivalently: for all $\{f_\mu, g_\mu\} \in S_2^*$ we have $\mathsf{b}_2\big(\{\Phi^{-1} f_\mu, \Phi^{-1} g_\mu\}\big) = \mathsf{b}_\mu(\{f_\mu, g_\mu\})$.

Set $\Theta = \Phi^{-1} W \Phi$, let $\{f_2, g_2\} \in S_2^*$ be arbitrary, and calculate

$$\begin{aligned} \mathsf{b}_2(\{f_2, g_2\}) &= \mathsf{b}_\mu(\{\Phi f_2, \Phi g_2\}) \\ &= \mathsf{V}^{-1}\mathsf{b}_\mu(\{W\Phi f_2, W\Phi g_2\}) \\ &= \mathsf{V}^{-1}\mathsf{b}_2\big(\{\Phi^{-1} W\Phi f_2, \Phi^{-1} W\Phi g_2\}\big) \\ &= \mathsf{V}^{-1}\mathsf{b}_2\big(\{\Theta f_2, \Theta g_2\}\big). \end{aligned}$$

Since $\mathsf{V}^{-1} = \Gamma^{-1}\Lambda$, (v) holds.

(v)⇒(iv). Assume (v) and define $W = \Phi\Theta\Phi^{-1}$. Then $W : \mathfrak{C}_\mu \to \mathfrak{C}_\mu$ is a linear bijection such that $WS_\mu = S_\mu W$ and $WS_\mu^* = S_\mu^* W$. By Theorem 2.9(i) there exists a unimodular $d \times d$ matrix polynomial $\mathcal{W}(z)$ and an invertible matrix $\mathsf{V}$ satisfying $\mathcal{W}(z)\mathcal{P}(z) \equiv \mathcal{P}(z)\mathsf{V}$ and such that $W$ is operator of multiplication by $\mathcal{W}(z)$.

By properties of $\Phi$ and $\Theta$ we have for all $\{f_\mu, g_\mu\} \in S_\mu^*$

$$\begin{aligned}
\mathsf{b}_\mu(\{Wf_\mu, Wg_\mu\}) &= \mathsf{b}_\mu\big(\{\Phi\Theta\Phi^{-1}f_\mu, \Phi\Theta\Phi^{-1}g_\mu\}\big)\\
&= \mathsf{b}_2\big(\{\Theta\Phi^{-1}f_\mu, \Theta\Phi^{-1}g_\mu\}\big)\\
&= \Lambda^{-1}\Gamma\mathsf{b}_2\big(\{\Phi^{-1}f_\mu, \Phi^{-1}g_\mu\}\big)\\
&= \Lambda^{-1}\Gamma\mathsf{b}_\mu(\{f_\mu, g_\mu\}).
\end{aligned}$$

By the definition of the boundary mapping $\mathsf{b}_\mu$ we have

$$\begin{aligned}
\mathcal{W}(z)\mathcal{P}(z)\mathsf{b}_\mu(\{f_\mu, g_\mu\}) &\equiv z\mathcal{W}(z)f_\mu(z) - \mathcal{W}(z)g_\mu(z),\\
&\equiv \mathcal{P}(z)\mathsf{b}_\mu(\{Wf_\mu, Wg_\mu\}),\\
&\equiv \mathcal{P}(z)\Lambda^{-1}\Gamma\mathsf{b}_\mu(\{f_\mu, g_\mu\}).
\end{aligned}$$

Thus

$$\mathcal{W}(z)\mathcal{P}(z)\mathsf{b}_\mu(\{f_\mu, g_\mu\}) \equiv \mathcal{P}(z)\Lambda^{-1}\Gamma\mathsf{b}_\mu(\{f_\mu, g_\mu\}).$$

The fact that $\mathsf{b}_\mu$ is a surjection implies that $\mathcal{W}(z)\mathcal{P}(z) \equiv \mathcal{P}(z)\Lambda^{-1}\Gamma$. This proves (iv).

To prove the last claim in the theorem, assume (iv). Then, by Lemma 2.8 with $\mathsf{V} = \Gamma^{-1}\Lambda$, there exists a $d \times d$ invertible matrix $\mathsf{W}$ such that $\mathsf{W}\mathsf{P}_\infty = \mathsf{P}_\infty\Gamma^{-1}\Lambda$. Therefore,

$$\operatorname{nul}\big(\mathsf{P}_\infty\Lambda^{-1}\big) = \operatorname{nul}\big(\mathsf{P}_\infty\Gamma^{-1}\big).$$

Now, $T_{\widetilde{A}_\Lambda}(\infty) = T_{\widetilde{A}_\Gamma}(\infty)$ follows from Theorem 4.4. □

**Author Contributions** B.Ć. and A.D. wrote the manuscript together. All authors reviewed the manuscript.

**Funding** No funding was received.

**Data Availibility** No datasets were generated or analysed during the current study.

## Declarations

**Competing interests** The authors declare no competing interests.

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