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Convex Interval Hull of Finite Sets in Real Linear Spaces: Extreme Points and Unbounded Images

Branko Ćurgus
Department of Mathematics, Western Washington University, Bellingham, U.S.A.
curgus@wwu.edu

Krzysztof Kołodziejczyk
Faculty of Pure and Applied Mathematics, Wrocław University of Science and Technology, Poland
krzysztof.kolodziejczyk@pwr.edu.pl

In memory of John R. Reay

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Let S be a finite set in a real linear space and let $\mathcal{J}_S$ be a family consisting of $|S|$ intervals in $\mathbb{R}$. In this paper we deal with a convex operator $\mathrm{co}(S, \mathcal{J}_S)$ called the convex interval hull. This operator generalizes the familiar concepts of the convex hull, $\mathrm{conv}(S)$, and the affine hull, $\mathrm{aff}(S)$, of S. The set $\mathrm{co}(S, \mathcal{J}_S)$ is a convex subset of the linear space and can be either bounded or unbounded, depending on the families $\mathcal{J}_S$. In this paper we apply $\mathrm{co}(S, \mathcal{J}_S)$ to obtain unbounded images of a finite set S. As special images of $\mathrm{co}(S, \mathcal{J}_S)$ for finite S we obtain such unbounded objects as: hyperplanes, cylinders, cones, penumbras and wedges. We also apply $\mathrm{co}(S, \mathcal{J}_S)$ to study some properties of extreme points. In relation to $\mathrm{co}(S, \mathcal{J}_S)$ we introduce the so-called extreme interval operator $\mathrm{Eco}(S)$ and prove some analogues of the celebrated Minkowski-Krein-Milman's theorem.

Keywords: Convex set, cone, convex interval hull, extreme point, extreme interval operator, Minkowski-Krein-Milman property.

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1. Introduction

In this note $\mathbf{L}$ denotes a real linear space. Let $S = \{x_1, \ldots, x_m\} \subset \mathbf{L}$ be a finite set of distinct points in $\mathbf{L}$. By $S_{\hat{i}}$ we denote the set $S \setminus \{x_i\}$. If $\mathcal{A}$ is a family of sets in $\mathbf{L}$, then by $\bigcap \mathcal{A}$ and $\bigcup \mathcal{A}$ we denote the intersection and the union, respectively, of all elements of $\mathcal{A}$. For $\alpha \in \mathbb{R}$ by $\alpha\mathcal{A}$ we denote the family of sets $\alpha A = \{\alpha x : x \in A\}$, where $A \in \mathcal{A}$. For subsets K and M of $\mathbf{L}$ and real numbers α and β we put

$$\alpha K \pm \beta M = \{\alpha x \pm \beta y \ : \ x \in K, \ y \in M\}.$$

For this type of operations on sets and many more much general ones we refer the reader to [6].

In the (ordinary) convexity theory and also in any of its generalizations the concept of a convex hull operator plays a central role. One is aware that convexity in a linear space is defined in terms of an algebraic structure. Hence, it is natural that

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the convex hull operator also has some algebraic structure. It is not a surprise that also any change in the algebraic requirements in the definition of the convex hull results in some generalizations. We show here that one such change results in a generalization which gives an unified approach to several known concepts.

The very classical convex sets associated with the finite set S are the convex hull of S and the affine hull of S defined by

$$\operatorname{conv}(S) = \left\{ \sum_{j=1}^{m} \alpha_j \, x_j \; : \; x_j \in S, \; \alpha_j \geq 0, \; \sum_{j=1}^{m} \alpha_j = 1 \right\}$$

and

$$\operatorname{aff}(S) = \left\{ \sum_{j=1}^{m} \alpha_j \, x_j \; : \; x_j \in S, \; \alpha_j \in \mathbb{R}, \; \sum_{j=1}^{m} \alpha_j = 1 \right\} \tag{1}$$

respectively. The operators $S \mapsto \operatorname{conv}(S)$ and $S \mapsto \operatorname{aff}(S)$ belong to the class of relatively recently introduced operator $S \mapsto \operatorname{co}(S, \mathcal{J}_S)$ defined in [3] by the following definition.

Definition 1.1. Let $S = \{x_1, \ldots, x_m\}$ be a finite set of distinct points in a linear space $\mathbf{L}$ and let $\mathcal{J}_S = \{I_1, \ldots, I_m\}$, $I_j \subset \mathbb{R}$, $j = 1, \ldots, m$, be a family of nonempty intervals (some of which can be degenerated to a singleton). By $\operatorname{co}(S, \mathcal{J}_S)$ we denote the set defined by

$$\operatorname{co}(S, \mathcal{J}_S) := \left\{ \sum_{j=1}^{m} \alpha_j \, x_j \; : \; x_j \in S, \; \alpha_j \in I_j, \; \sum_{j=1}^{m} \alpha_j = 1 \right\}. \tag{2}$$

The set $\operatorname{co}(S, \mathcal{J}_S)$ is called the *convex interval hull of S*.

Obviously, for $\mathcal{J}_S = \{I_1, \ldots, I_m\}$ with $I_j = [0, t_j]$, where $t_j \geq 1$, $j = 1, \ldots, m$, we have $\operatorname{co}(S, \mathcal{J}_S) = \operatorname{conv}(S)$ and for $\mathcal{J}_S = \{I_1, \ldots, I_m\}$ with $I_j = \mathbb{R}$, $j = 1, \ldots, m$, we have $\operatorname{co}(S, \mathcal{J}_S) = \operatorname{aff}(S)$.

Notice that one could go even further and instead of (2) define for any real number t the set $\operatorname{co}(S, \mathcal{J}_S, t)$ as follows

$$\operatorname{co}(S, \mathcal{J}_S, t) := \left\{ \sum_{j=1}^{m} \alpha_j \, x_j \; : \; x_j \in S, \; \alpha_j \in I_j, \; \sum_{j=1}^{m} \alpha_j = t \right\}.$$

The latter definition, however, is only slightly more general than the previous one, because in all cases but $t = 0$ the set $\operatorname{co}(S, \mathcal{J}_S, t)$ can be expressed by means of $\operatorname{co}(S, \mathcal{J}_S)$.

Proposition 1.2. *Let t be a nonzero real number. Then*

$$\operatorname{co}(S, \mathcal{J}_S, t) = \operatorname{co}\big(t\, S, (1/t)\mathcal{J}_S\big).$$

The notion $\operatorname{co}(S, \mathcal{J}_S, 0)$ is still of some interest, particularly when every interval in $\mathcal{J}_S$ is $\mathbb{R}$. It can be easily checked that in this case $\operatorname{co}(S, \mathcal{J}_S, 0)$ represents the linear subspace L parallel to $\operatorname{aff}(S)$. On several occasions it will be convenient to use this special set in our note, for instance in Theorems 2.3 and 2.10.

We have already mentioned two unbounded convex interval hulls assigned to a finite set in $\mathbf{L}$, namely $\operatorname{co}(S, \mathcal{J}_S)$ and $\operatorname{co}(S, \mathcal{J}_S, 0)$, both when every interval in $\mathcal{J}_S$ is the entire real line. In Section 2 we explore possibilities of obtaining unbounded sets further. We show that cones, halfspaces, cylinders, wedges, penumbras and some other sets can be obtained as images of finite sets in a linear space by the convex interval hull. In addition to the operator $S \mapsto \operatorname{co}(S, \mathcal{J}_S)$ we will also consider the operator $S \mapsto \operatorname{Eco}(S)$ that assigns to S its extreme points with respect to the convex interval hull $\operatorname{co}(S, \mathcal{J}_S)$. In Section 3 we examine relationships between extreme points (defined with respect to the operator conv) and extreme points with respect to the operator $\operatorname{co}(S, \mathcal{J}_S)$. We also obtain Minkowski-Krein-Milman – type theorems.

2. Unbounded images of convex interval hulls

Since our primary interest in this paper is to study the cases which lead to unbounded sets $\operatorname{co}(S, \mathcal{J}_S)$, it is convenient to recall the following theorem from [3, Theorem 3.4].

Theorem 2.1. *Let $S = \{x_1, \dots, x_m\}$ be a finite set of distinct points in a linear space $\mathbf{L}$ and let $\mathcal{J}_S = \{I_1, \dots, I_m\}$, $I_j \subseteq \mathbb{R}$, $j = 1, \dots, m$, be a family of nonempty intervals. The set $\operatorname{co}(S, \mathcal{J}_S)$ is bounded if and only if at least one of the conditions below is satisfied.*

(1) *All the intervals in $\mathcal{J}_S$ are bounded below.*

(2) *All the intervals in $\mathcal{J}_S$ are bounded above.*

(3) *At most one interval in $\mathcal{J}_S$ is unbounded.*

If any of the conditions (1)–(3) *is satisfied, then there exists a family of bounded intervals $\mathcal{J}'_S$ such that $\operatorname{co}(S, \mathcal{J}_S) = \operatorname{co}(S, \mathcal{J}'_S)$.*

Let A be a set and let x_0 be a point in $\mathbf{L}$. By $\operatorname{cone}(x_0, A)$ we denote the *cone* with vertex x_0 and directrix $\operatorname{conv}(A)$ which is the convex set defined by

$$\operatorname{cone}(x_0, A) = \big\{(1-t)x_0 + tv \ : \ v \in \operatorname{conv}(A),\ t \geq 0\big\}. \tag{3}$$

If $A = \{y\}$ is a singleton, then $\operatorname{cone}(x_0, \{y\})$ is called a *ray* in $\mathbf{L}$.

Theorem 2.2. *Let $S = \{x_1, \dots, x_m\}$ be a finite set of distinct points in a linear space $\mathbf{L}$ and let $\mathcal{J}_S = \{I_1, \dots, I_m\}$, $I_j \subseteq \mathbb{R}$, $j = 1, \dots, m$, be a family of nonempty intervals. Then $\operatorname{co}(S, \mathcal{J}_S)$ is unbounded if and only if there exists a ray $\mathcal{R}$ contained in $\operatorname{co}(S, \mathcal{J}_S)$.*

Proof. The sufficiency is obvious. Now we prove the necessity. Assume that $\operatorname{co}(S, \mathcal{J}_S)$ is unbounded and hence nonempty. Thus there exists

$$y = \sum_{j=1}^{m} \gamma_j x_j \in \operatorname{co}(S, \mathcal{J}_S).$$

From Theorem 2.1 it follows that among the sets in $\mathcal{J}_S$ there are two, say I_1 and I_2, which contain, respectively, half-lines of the type $(-\infty, b]$ and $[a, \infty)$. Apparently, $(-\infty, \gamma_1] \subset I_1$ and $[\gamma_2, \infty) \subset I_2$ and for any $t \geq 0$ we have $\gamma_1 - t \in I_1$ and $\gamma_2 + t \in I_2$.

Let
$$\begin{aligned}\mathcal{R} &= \{\, y_t = y + t(x_2 - x_1) \ : \ t \geq 0\} \\ &= \{\, y_t = (\gamma_1 - t)x_1 + (\gamma_2 + t)x_2 + \gamma_3 x_3 + \cdots + \gamma_m x_m \ : \ t \geq 0\}.\end{aligned}$$

Clearly, $\mathcal{R}$ is a ray emanating from y and parallel to $x_2 - x_1$ and is contained in $\mathrm{co}(S, \mathcal{J}_S)$. The proof is complete. □

Take a non-empty convex set $M \subset \mathbf{L}$ and a subspace L of $\mathbf{L}$. Following [10] the set $M + L$ is called an *F-cylinder* in $\mathbf{L}$.

Theorem 2.3. *Let k and m be positive integers with $m - k \geq 2$. In $\mathbf{L}$ consider finite sets $S_1 = \{x_1, \ldots, x_k\}$, $S_2 = \{x_{k+1}, \ldots, x_m\}$ and put $S = S_1 \cup S_2$. Further, let $\mathcal{J}_S = \{I_1, \ldots, I_k, I_{k+1}, \ldots, I_m\}$ be a family of closed intervals in $\mathbb{R}$ (possibly singletons) such that $I_j = [a_j, b_j]$, $a_j \leq b_j$, for $j = 1, \ldots, k$ and $I_j = \mathbb{R}$ for $j = k+1, \ldots, m$. Then there exist a family $\mathcal{J}'_S$ and a subspace $L = \mathrm{co}(S_2, \mathcal{J}_{S_2}, 0)$ of $\mathbf{L}$ such that $\mathrm{co}(S, \mathcal{J}_S)$ is an F-cylinder of the form*

$$\mathrm{co}(S, \mathcal{J}_S) = \mathrm{co}(S, \mathcal{J}'_S) + \mathrm{co}(S_2, \mathcal{J}_{S_2}, 0).$$

Proof. Put $a = \sum_{j=1}^{k} a_j$ and $b = \sum_{j=1}^{k} b_j$. Clearly $a \leq b$. Take any real numbers a'_j, b'_j, $j = k+1, \ldots, m$, such that

$$a'_j \leq b'_j, \qquad \sum_{j=k+1}^{m} a'_j = 1 - b, \qquad \sum_{j=k+1}^{m} b'_j = 1 - a.$$

Notice that such numbers do exist. For instance, the numbers

$$a'_j = \frac{1-b}{m-k}, \quad b'_j = \frac{1-a}{m-k}, \quad j = k+1, \ldots, m \tag{4}$$

satisfy the above requirements. Define a new family $\mathcal{J}'_S$ of closed and bounded intervals by

$$\mathcal{J}'_S = \{I_1, \ldots, I_k, I'_{k+1}, \ldots, I'_m\}, \quad I'_j = \left[a'_j, b'_j\right], \quad j = k+1, \ldots, m.$$

We will show that $\mathrm{co}(S, \mathcal{J}_S)$ is an F-cylinder of the form

$$\mathrm{co}(S, \mathcal{J}_S) = \mathrm{co}(S, \mathcal{J}'_S) + L = \mathrm{co}(S, \mathcal{J}'_S) + \mathrm{co}(S_2, \mathcal{J}_{S_2}, 0). \tag{5}$$

To prove equality (5) take any $x \in \mathrm{co}(S, \mathcal{J}_S)$. Then $x = \sum_{j=1}^{m} c_j x_j$ with $c_j \in I_j$ and $\sum_{j=1}^{m} c_j = 1$. Since

$$\sum_{j=k+1}^{m} c_j = 1 - \sum_{j=1}^{k} c_j \in \left[1-b, 1-a\right],$$

we can choose any numbers $\xi_j \in I'_j$, $j = k+1, \ldots, m$, such that

$$\sum_{j=k+1}^{m} c_j = \sum_{j=k+1}^{m} \xi_j. \tag{6}$$

Referring to the special case (4), we could choose

$$\xi_j = \frac{1}{m-k} \sum_{j=k+1}^{m} c_j \in \left[\frac{1-b}{m-k}, \frac{1-a}{m-k}\right], \quad j = k+1, \ldots, m.$$

Then $$x = \sum_{j=1}^{m} c_j x_j = \sum_{j=1}^{k} c_j x_j + \sum_{j=k+1}^{m} \xi_j x_j + \Big(x_m + \sum_{j=k+1}^{m} (c_j - \xi_j) x_j\Big) - x_m.$$

Using (6) we see that $\sum_{j=1}^{k} c_j+\sum_{j=k+1}^{m} \xi_j = 1$ and $1+\sum_{j=k+1}^{m}(c_j-\xi_j) = 1$. Therefore by (2) and (1) we have respectively

$$\sum_{j=1}^{k} c_j x_j + \sum_{j=k+1}^{m} \xi_j x_j \in \operatorname{co}(S, \mathcal{J}'_S)$$

and

$$x_m + \sum_{j=k+1}^{m} (c_j - \xi_j) x_j \in \operatorname{aff}(S_2).$$

Thus, we proved that for an arbitrary $x \in \operatorname{co}(S, \mathcal{J}_S)$ it holds that

$$x \in \operatorname{co}(S, \mathcal{J}'_S) + (\operatorname{aff}(S_2) - x_m) = \operatorname{co}(S, \mathcal{J}'_S) + L = \operatorname{co}(S, \mathcal{J}'_S) + \operatorname{co}(S_2, \mathcal{J}_{S_2}, 0).$$

A proof of the converse inclusion in (5) is straightforward. □

Theorem 2.4. *Let k and m be positive integers such that $m - k \geq 1$ and $\delta \in \mathbb{R}$. Further, let $S_1 = \{x_1, \ldots, x_k\}$ and $S_2 = \{x_{k+1}, \ldots, x_m\}$ be finite sets in* **L** *and put $S = S_1 \cup S_2$. Consider the family $\mathcal{J}_S = \{I_1, ..., I_k, I_{k+1}, ..., I_m\}$, where $I_j = (-\infty, \delta]$ for $j = 1, \ldots, k$, and $I_j = [0, +\infty)$ for $j = k+1, \ldots, m$. Then*

$$\operatorname{co}(S, \mathcal{J}_S) = \{x_\delta\} + \operatorname{cone}\big(0, S_2 - S_1\big), \tag{7}$$

where

$$x_\delta = \begin{cases} \displaystyle\sum_{i=1}^{k} \frac{1}{k} x_i & \text{when } \delta \leq \dfrac{1}{k}, \\ \displaystyle\sum_{i=1}^{k} \delta x_i - \sum_{j=k+1}^{m} \frac{\delta k - 1}{m-k} x_j & \text{when } \delta > \dfrac{1}{k}. \end{cases}$$

Proof. Before proving (7), notice that $v \in \operatorname{cone}\big(0, S_2 - S_1\big)$ if and only if

$$v = \sum_{j=k+1}^{m} \xi_j x_j - \sum_{j=1}^{k} \xi_j x_j, \quad \text{where} \quad \sum_{j=k+1}^{m} \xi_j = \sum_{j=1}^{k} \xi_j, \ \xi_j \geq 0, j = 1, \ldots, m.$$

To show this we first observe that any linear combination (in particular any convex combination which we in fact need) of the $k(m-k)$ vectors from $S_2 - S_1$ can be written as follows

$$\begin{aligned} \sum_{j=1}^{k} \sum_{i=k+1}^{m} \alpha_{ij}(x_i - x_j) &= \sum_{i=k+1}^{m} \Big(\sum_{j=1}^{k} \alpha_{ij}\Big) x_i - \sum_{j=1}^{k} \Big(\sum_{i=k+1}^{m} \alpha_{ij}\Big) x_j \\ &= \sum_{i=k+1}^{m} \eta_i x_i - \sum_{j=1}^{k} \eta_j x_j, \end{aligned}$$

where

$$\sum_{i=k+1}^{m} \eta_i = \sum_{i=k+1}^{m} \Big(\sum_{j=1}^{k} \alpha_{ij}\Big) = \sum_{j=1}^{k} \Big(\sum_{i=k+1}^{m} \alpha_{ij}\Big) = \sum_{j=1}^{k} \eta_j \tag{8}$$

and all the numbers in (8) for a convex combination are nonnegative. This together with (3) justifies the characterization of any $v \in \operatorname{cone}\big(0, S_2 - S_1\big)$ given at the beginning of the proof.

To show equality (7), let $x \in \mathrm{co}(S, \mathcal{J}_S)$ be such that

$$x = \sum_{l=1}^{m} c_l x_l, \quad \sum_{l=1}^{m} c_l = 1, \quad c_l \in I_l, \quad l = 1, \dots, m.$$

We start with the case $\delta > \frac{1}{k}$. Express x as follows

$$\begin{aligned} x &= \sum_{l=1}^{k} \delta x_l + \sum_{l=k+1}^{m} c_l x_l + \sum_{l=1}^{k} (c_l - \delta) x_l \\ &= \sum_{l=1}^{k} \delta x_l + \sum_{l=k+1}^{m} c_l x_l - \sum_{l=1}^{k} (\delta - c_l) x_l \end{aligned}$$

Let α be a nonnegative number. Clearly we have

$$x = \sum_{l=1}^{k} \delta x_l - \sum_{l=k+1}^{m} \alpha x_l + \sum_{l=k+1}^{m} (c_l + \alpha) x_l - \sum_{l=1}^{k} (\delta - c_l) x_l. \tag{9}$$

Now we will check whether there exists an α for which the difference of the last two sums in (9) represents a vector from $\mathrm{cone}(S_2 - S_1)$. To use the characterization given at the beginning of the proof we first observe that all the numbers $c_l + \alpha$ and $\delta - c_l$ in (9) are nonnegative. In addition we need to find an α for which we have

$$\sum_{l=k+1}^{m} (c_l + \alpha) = \sum_{l=1}^{k} (\delta - c_l).$$

This equality is equivalent to

$$\sum_{l=k+1}^{m} c_l + (m-k)\alpha = k\delta - \sum_{l=1}^{k} c_l = k\delta - \Big(1 - \sum_{l=k+1}^{m} c_l\Big) = k\delta - 1 + \sum_{l=k+1}^{m} c_l$$

and therefore also to $(m-k)\alpha = k\delta - 1$. Thus

$$\alpha = \frac{k\delta - 1}{m - k} \geq 0.$$

When we substitute this value for α in the second sum in (9) we see that (9) guarantees validity of the inclusion

$$\mathrm{co}(S, \mathcal{J}_S) \subset \{x_\delta\} + \mathrm{cone}(S_2 - S_1)$$

in the case $\delta > \frac{1}{k}$.

Now suppose that $\delta \leq \frac{1}{k}$. Clearly, $c_l \leq \delta \leq \frac{1}{k}$ for $l = 1, \dots, k$. In this case write x in the form

$$x = \sum_{l=1}^{k} \frac{1}{k} x_l + \sum_{l=k+1}^{m} c_l x_l - \sum_{l=1}^{k} \Big(\frac{1}{k} - c_l\Big) x_l \tag{10}$$

and check that the difference of the last two sums in (10) also represents a vector from $\mathrm{cone}(S_2 - S_1)$. This implies that the same inclusion is true in this case. The proof of the converse inclusion in (7) in both cases is straightforward. □

Let A_1 and A_2 be disjoint subsets of $\mathbf{L}$. Following [7] we define the *penumbra* of A_2 with respect to A_1 as the set

$$P_{A_1}(A_2) = \{(1-\lambda)A_1 + \lambda A_2 \ : \ \lambda \geq 1\}. \tag{11}$$

If A_1 and A_2 are convex, then the penumbra $P_{A_1}(A_2)$ is also convex. With this notion we can have another look at the case $\delta = 0$ in Theorem 2.4.

Theorem 2.5. *Let $S_1 = \{x_1, \ldots, x_k\}$ and $S_2 = \{x_{k+1}, \ldots, x_m\}$ be subsets of $\mathbf{L}$ with disjoint convex hulls. Let $S = S_1 \cup S_2$. Let $\mathcal{J}_S = \{I_1, \ldots, I_k, I_{k+1}, \ldots, I_m\}$ be a family of closed intervals such that $I_j = (-\infty, 0]$, for $j = 1, \ldots, k$ and $I_j = [0, +\infty)$, $j = k+1, ..., m$. Then $\operatorname{co}(S, \mathcal{J}_S)$ is the penumbra of $\operatorname{conv}(S_2)$ with respect to $\operatorname{conv}(S_1)$*

$$\operatorname{co}(S, \mathcal{J}_S) = P_{\operatorname{conv}(S_1)}\big(\operatorname{conv}(S_2)\big). \tag{12}$$

Proof. To show (12), let $x \in \operatorname{co}(S, \mathcal{J}_S)$ be such that

$$x = \sum_{l=1}^{m} c_l x_l, \quad \sum_{l=1}^{m} c_l = 1, \quad c_i \leq 0, \quad i = 1, \ldots, k, \quad c_j \geq 0, \quad j = k+1, \ldots, m.$$

Let $\gamma = \sum_{i=1}^{k} c_i$, $\gamma \leq 0$. If $\gamma = 0$, then $c_i = 0$ for $i = 1, \ldots, k$ and

$$x \in \operatorname{conv}(S_2) \subset P_{\operatorname{conv}(S_1)}\big(\operatorname{conv}(S_2)\big). \tag{13}$$

Let $\gamma < 0$. To avoid a repetition of similar calculations we express x in two forms by making the indicated substitutions

$$\begin{aligned} x &= \gamma \sum_{i=1}^{k} \frac{c_i}{\gamma} x_i + (1-\gamma) \sum_{j=k+1}^{m} \frac{c_j}{1-\gamma} x_j \\ &= \sum_{j=k+1}^{m} \frac{c_j}{1+t} x_j + t \sum_{j=k+1}^{m} \frac{c_j}{1+t} x_j - t \sum_{i=1}^{k} \Big(-\frac{c_i}{t}\Big) x_i, \quad t = -\gamma > 0, \qquad (14) \\ &= (1-\lambda) \sum_{i=1}^{k} \frac{c_i}{1-\lambda} x_i + \lambda \sum_{j=k+1}^{m} \frac{c_j}{\lambda} x_j, \quad \lambda = 1-\gamma > 1. \qquad (15) \end{aligned}$$

The reader can easily check that from (15) (the case $\lambda > 1$) and (13) (the case $\lambda = 1$) it follows that x belongs to the penumbra. The proof of the converse inclusion in (12) is straightforward. □

Remark 2.6. On our way to establish (12) we obtained (14). Using the characterization from the beginning of the proof of Theorem 2.4 one can recognize in (14) the representation of a vector from $\operatorname{conv}(S_2) + \operatorname{cone}\big(0, S_2 - S_1\big)$. Moreover it is easy to check that when $\delta \leq 0$ in Theorem 2.4, then the convex interval hull $\operatorname{co}(S, \mathcal{J}_S)$ also satisfies the following equality

$$\operatorname{co}(S, \mathcal{J}_S) = \operatorname{conv}(S_1) + \operatorname{cone}\big(0, S_2 - S_1\big). \tag{16}$$

To see that this is true write x from $\operatorname{co}(S, \mathcal{J}_S)$ in the form

$$x = \sum_{j=1}^{k} c'_j x_j + \sum_{j=k+1}^{m} c_j x_j - \sum_{j=1}^{k} (c'_j - c_j) x_j,$$

where $$\sum_{j=1}^{k} c'_j = 1 \quad \text{and} \quad c'_j \geq 0, \quad j = 1, \ldots, k$$

and use the characterization of vectors from $\operatorname{cone}\big(0, S_2 - S_1\big)$ which was already mentioned in this remark. Using (12), (14) and (16) we see that the convex interval hull from Theorem 2.4 in the case $\delta = 0$ can be expressed in the following forms

$$\begin{aligned} \operatorname{co}(S, \mathcal{J}_S) &= \operatorname{conv}(S_2) + \operatorname{cone}\big(0, S_2 - S_1\big) = \operatorname{conv}(S_1) + \operatorname{cone}\big(0, S_2 - S_1\big) \\ &= P_{\operatorname{conv}(S_1)}\big(\operatorname{conv}(S_2)\big) \end{aligned}$$

Theorem 2.7. *Let $S = \{x_1, \ldots, x_m\}$, $m \geq 2$, be a set of distinct points in a linear space $\mathbf{L}$. Further, let $S_0 = \{x_0\} \cup S$ for some $x_0 \notin \operatorname{aff}(S) = F$. Denote by $F_0 = \operatorname{aff}(S_0)$ and by F_0^+ the affine halfspace bounded by F and containing x_0. Then for the family $\mathcal{J}_{S_0} = \{I_0, \ldots, I_m\}$, where $I_0 = [0, +\infty)$ and $I_1 = \cdots = I_m = \mathbb{R}$ we have*

$$\operatorname{co}(S_0, \mathcal{J}_{S_0}) = F_0^+.$$

Similarly, for any flat (in particular for a subspace of $\mathbf{L}$) $H = F + \{u\}$, $u \in \mathbf{L}$, and $H_0^+ = F_0^+ + \{u\}$ we have

$$\operatorname{co}(S'_0, \mathcal{J}_{S'_0}) = H_0^+,$$

where $S'_0 = S_0 + \{u\}$ and $\mathcal{J}_{S'_0} = \mathcal{J}_{S_0}$.

Proof. The first part of the theorem simply follows from Definition 1.1 and (1). The second part of the theorem is a consequence of the following observation which can be easily checked.

$$\operatorname{co}(S'_0, \mathcal{J}_{S'_0}) = \operatorname{co}(S_0 + \{u\}, \mathcal{J}_{S_0}) = \operatorname{co}(S_0, \mathcal{J}_{S_0}) + \{u\} = F_0^+ + \{u\} = H_0^+. \qquad \square$$

The next theorem shows a way in which a convex cone over a finite set in a linear space can be spanned. Notice that for the sake of consistency with next two results, in Theorem 2.8 we take $I_1 = \mathbb{R}$ instead of $I_1 = (-\infty, 1]$ which would be enough.

Theorem 2.8. *Let $S = \{x_1, \ldots, x_m\}$, $m \geq 2$, be a set of distinct points in a linear space $\mathbf{L}$. Put $I_1 = \mathbb{R}$, $I_2 = \cdots = I_m = [0, +\infty)$ and let $\mathcal{J}_S = \{I_1, I_2, \ldots, I_m\}$. Then*

$$\operatorname{co}(S, \mathcal{J}_S) = \operatorname{cone}(x_1, S_{\hat{1}}). \tag{17}$$

Proof. To show the inclusion $\operatorname{co}(S, \mathcal{J}_S) \subset \operatorname{cone}(x_1, S_{\hat{1}})$ take $z \in \operatorname{co}(S, \mathcal{J}_S)$. Then for some numbers $a_i \in I_i$ we have $z = \sum_{i=1}^m a_i x_i$. Let $t = \sum_{i=2}^m a_i$. The case $t = 0$ is possible only when $a_2 = \cdots = a_m = 0$ and in this case $z = x_1 \in \operatorname{cone}(x_1, S_{\hat{1}})$. If $t > 0$, then $a_1 = 1 - t \in I_1$ and

$$z = \sum_{i=1}^{m} a_i x_i = a_1 x_1 + t \sum_{i=2}^{m} \frac{a_i}{t} x_i = (1 - t) x_1 + t \sum_{i=2}^{m} \frac{a_i}{t} x_i.$$

Clearly $$\sum_{i=2}^{m} \frac{a_i}{t} x_i \in \operatorname{conv}(S_{\hat{1}})$$

and in view of (3) $z \in \operatorname{cone}(x_1, S_{\hat{1}})$. The converse inclusion in (17) is immediate. $\square$

Theorem 2.8 in conjunction with some known properties of convex sets imply the following corollary.

Corollary 2.9. *Assume that k and m are positive integers such that $m-k \geq 1$. Let $S_1 = \{x_1, \ldots, x_k\}$ and $S_2 = \{x_{k+1}, \ldots, x_m\}$ be finite sets in $\mathbf{L}$ and $S = S_1 \cup S_2$. Let $\mathcal{J}_S = \{I_1, \ldots, I_k, I_{k+1}, \ldots, I_m\}$ be a family of closed intervals such that $I_j = \mathbb{R}$, for $j = 1, \ldots, k$ and $I_j = [0, \infty)$, $j = k+1, \ldots, m$. Then*

$$\operatorname{conv}\left(\bigcup \left\{ \operatorname{cone}\bigl(x_i, S_2\bigr) : i = 1, \ldots, k \right\}\right) \subset \operatorname{co}(S, \mathcal{J}_S). \tag{18}$$

The following theorem shows that for S and $\mathcal{J}_S$ as in Corollary 2.9, the set $\operatorname{co}(S, \mathcal{J}_S)$ is much bigger than the set on the left-hand side in (18).

Theorem 2.10. *Assume that k and m are positive integers such that $m-k \geq 1$. Let $S_1 = \{x_1, \ldots, x_k\}$ and $S_2 = \{x_{k+1}, \ldots, x_m\}$ be finite sets in $\mathbf{L}$ and $S = S_1 \cup S_2$. Let $\mathcal{J}_S = \{I_1, \ldots, I_k, I_{k+1}, \ldots, I_m\}$ be a family of closed intervals such that $I_j = \mathbb{R}$, for $j = 1, \ldots, k$ and $I_j = [0, \infty)$, $j = k+1, \ldots, m$. Then*

$$\operatorname{co}(S, \mathcal{J}_S) = \bigcup \left\{ \operatorname{cone}\bigl(y, S_2\bigr) : y \in \operatorname{aff}(S_1) \right\} \bigcup \bigl(\operatorname{co}(S_1, \mathcal{J}_{S_1}, 0) + \operatorname{conv}(S_2)\bigr) \tag{19}$$

Proof. We start in the same way as in the proof of Theorem 2.8. Let $z \in \operatorname{co}(S, \mathcal{J}_S)$. Then $z = \sum_{i=1}^{m} a_i x_i$ for some numbers $a_i \in I_i$ with $\sum_{i=1}^{m} a_i = 1$. Clearly

$$z = \sum_{i=1}^{m} a_i x_i = \sum_{i=1}^{k} a_i x_i + \sum_{i=k+1}^{m} a_i x_i. \tag{20}$$

Let $t = \sum_{i=k+1}^{m} a_i \geq 0$. If $t = 0$, then $a_i = 0$ for $i = k+1, \ldots, m$ and $\sum_{i=1}^{k} a_i = 1$. In this case we have

$$z = \sum_{i=1}^{k} a_i x_i \in \operatorname{co}(S_1, \mathcal{J}_{S_1}) \subset \operatorname{aff}(S_1) \subset \bigcup \left\{ \operatorname{cone}\bigl(y, S_2\bigr) : y \in \operatorname{aff}(S_1) \right\}.$$

If $t > 0$ and $t \neq 1$, then

$$z = \sum_{i=1}^{m} a_i x_i = (1-t) \sum_{i=1}^{k} \frac{a_i}{t-1} x_i + t \sum_{i=k+1}^{m} \frac{a_i}{t} x_i$$

One can easily check that the vector $y = \sum_{i=1}^{k} \frac{a_i}{t-1} x_i$ belongs to $\operatorname{aff}(S_1)$ and the vector $\sum_{i=k+1}^{m} \frac{a_i}{t} x_i$ belongs to $\operatorname{conv}(S_2)$. In view of (3)

$$z \in \bigcup \left\{ \operatorname{cone}\bigl(y, S_2\bigr) : y \in \operatorname{aff}(S_1) \right\}.$$

Now we consider the case when $t = 1$. Clearly, the first sum in (20) represents a vector from $\operatorname{co}(S_1, \mathcal{J}_{S_1}, 0)$ and the second sum a vector from $\operatorname{conv}(S_2)$. This ends the proof of the considered inclusion. The converse inclusion in (19) is immediate. □

We will need the following lemma.

Lemma 2.11. *Let $A \subset B = \{x_1, ..., x_m\} \subset \mathbf{L}$ and let $\mathcal{J}_A = \{I_1^A, I_2^A, ..., I_n^A\}$ and $\mathcal{J}_B = \{I_1^B, I_2^B, \dots, I_m^B\}$, $n \le m$, be families of intervals such that $I_i^A \subset I_i^B$ when $x_i \in A$ and $0 \in I_j^B$ when $x_j \in B \setminus A$. Then*

$$\operatorname{co}(A, \mathcal{J}_A) \subset \operatorname{co}(B, \mathcal{J}_B).$$

Notice that when we put $A = S_{\hat{1}}$ and $B = S$ then Example 3.7 shows that the assumption that $0 \in I_j^B$ when $x_j \in B \setminus A$ is essential in Lemma 2.11. Indeed, in that example $0 \notin I_1$ and $\operatorname{co}(S_{\hat{1}}, \mathcal{J}_{\hat{1}})$ is not a subset of $\operatorname{co}(S, \mathcal{J}_S)$.

Theorem 2.12. *Let k and m be positive integers satisfying $m-k \ge 1$. Assume that $S_1 = \{x_1, \dots, x_k\}$ and $S_2 = \{x_{k+1}, \dots, x_m\}$ are finite sets in $\mathbf{L}$ and $S = S_1 \cup S_2$. Let $\mathcal{J}_S = \{I_1, \dots, I_k, I_{k+1}, \dots, I_m\}$ be a family of closed intervals such that $I_j = [0, +\infty)$, for $j = 1, \dots, k$ and $I_j = \mathbb{R}$, $j = k+1, \dots, m$. Put*

$$S_i^* = \{x_i\} \cup S_2, \quad \mathcal{J}_{S_i^*} = \{I_i, I_{k+1}, \dots, I_m\}, \quad i = 1, \dots, k.$$

Then $\operatorname{co}(S, \mathcal{J}_S)$ is a wedge described in the following way

$$\operatorname{co}(S, \mathcal{J}_S) = \operatorname{conv}\Big(\bigcup_{i=1}^{k} \operatorname{co}(S_i^*, \mathcal{J}_{S_i^*})\Big). \tag{21}$$

Proof. In order to establish the inclusion $\operatorname{co}(S, \mathcal{J}_S) \subset \operatorname{conv}\big(\bigcup_{i=1}^{k} \operatorname{co}(S_i^*, \mathcal{J}_{S_i^*})\big)$ it is enough to show, see [7, 11], that any $z \in \operatorname{co}(S, \mathcal{J}_S)$ is a convex combination of some points $z_i \in \operatorname{co}(S_i^*, \mathcal{J}_{S_i^*})$, $i = 1, \dots, k$. Take $z \in \operatorname{co}(S, \mathcal{J}_S)$. Thus

$$z = \sum_{j=1}^{k} \lambda_j x_j + \sum_{j=k+1}^{m} \lambda_j x_j$$

were $\lambda_j \ge 0$ for $j = 1, \dots, k$ and $\lambda_j \in \mathbb{R}$ for $j = k+1, \dots, m$ with $\sum_{j=1}^{m} \lambda_j = 1$. If $\lambda_1 = \cdots = \lambda_k = 0$, then obviously $z \in \operatorname{conv}(S_2) \subset \operatorname{co}(S_i^*, \mathcal{J}_{S_i^*})$, $i = 1, \dots, k$, and the inclusion in (21) is satisfied. If at least one of $\lambda_1, \dots, \lambda_k$ is positive put $\lambda = \sum_{i=1}^{k} \lambda_i > 0$ and define points

$$z_i = \lambda x_i + \sum_{j=k+1}^{m} \lambda_j x_j, \quad i = 1, \dots, k.$$

Clearly, $z_i \in \operatorname{co}(S_i^*, \mathcal{J}_{S_i^*})$, $i = 1, \dots, k$. For $\alpha_i = \frac{\lambda_i}{\lambda}$ and the convex combination $\sum_{i=1}^{k} \alpha_i z_i$ we have

$$\begin{aligned}\sum_{i=1}^{k} \alpha_i z_i &= \sum_{i=1}^{k} \frac{\lambda_i}{\lambda}\Big(\lambda x_i + \sum_{j=k+1}^{m} \lambda_j x_j\Big) = \sum_{i=1}^{k} \lambda_i x_i + \sum_{i=1}^{k} \frac{\lambda_i}{\lambda} \sum_{j=k+1}^{m} \lambda_j x_j \\ &= \sum_{i=1}^{k} \lambda_i x_i + \sum_{j=k+1}^{m} \lambda_j x_j = z.\end{aligned}$$

Hence $z \in \operatorname{conv}\Big(\bigcup_{i=1}^{k} \operatorname{co}(S_i^*, \mathcal{J}_{S_i^*})\Big)$.

To establish the converse inclusion in (21) notice that by Lemma 2.11

$$\operatorname{co}(S_i^*, \mathcal{J}_{S_i^*}) \subset \operatorname{co}(S, \mathcal{J}_S)$$

for $i = 1, \ldots, k$. This together with convexity of $\operatorname{co}(S, \mathcal{J}_S)$ implies

$$\operatorname{conv}\left(\bigcup_{i=1}^{k} \operatorname{co}(S_i^*, \mathcal{J}_{S_i^*})\right) \subset \operatorname{co}(S, \mathcal{J}_S)$$

and ends the proof of (21). □

3. Extreme points

In many literature sources extreme points of sets in a linear space with ordinary convexity are considered for convex sets, cf. [2, 5, 9, 12, 13]. In [4] one can find the following definition. A point x is an *extreme point* of a set S in $\mathbf{L}$ if

$$x \in S \quad \text{but} \quad x \notin \operatorname{conv}(S \setminus \{x\}). \tag{22}$$

A consequence of this definition is the fact that x is an extreme point of S if and only if $\operatorname{conv}(S \setminus \{x\})$ is a proper (convex) subset of $\operatorname{conv}(S)$. Let $E(S)$ denote the *extreme set*, that is, the set of all extreme points of S.

Example 3.1. This and every other example was calculated and plotted using *Mathematica*. Points of S are listed starting from the lowest point that is furthermost to the left. Then we proceed counterclockwise, finishing with the point inside. In each figure the points in S are marked with black dots (•) and the polygon $\operatorname{co}\big(S, \mathcal{J}_S\big)$ is shaded gray with its edges slightly darker. Let

$$S = \big\{x_1 = (-1, 0), x_2 = (1, 0), x_3 = (0, \sqrt{3})\big\}$$

and

$$\mathcal{J}_S = \big\{I_1 = [0, 2/3], I_2 = [0, 2/3], I_3 = [0, 2/3]\big\}.$$

The set $\operatorname{co}(S, \mathcal{J}_S)$ is shown in Figure 1. Notice that the sets S and $\operatorname{co}(S, \mathcal{J}_S)$ are disjoint.

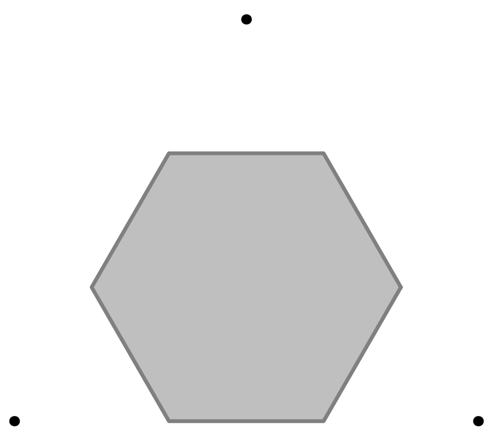

Figure 1: S disjoint with $\operatorname{co}(S, \mathcal{J}_S)$

Example 3.1 illustrates that in general it is possible to have $S \not\subset \operatorname{co}(S, \mathcal{J}_S)$. This observation is a good motivation to show the following proposition.

Proposition 3.2. *Let S be a finite set in $\mathbf{L}$ and let $\mathcal{J}_S$ be a family of intervals. If for each $x_i \in E(S)$ we have $\{0,1\} \subset I_i$ and $0 \in I_j$ for each $x_j \in S \setminus E(S)$, then* $\operatorname{conv}(S) \subset \operatorname{co}(S, \mathcal{J}_S)$.

Proof. Our assumptions guarantee that $E(S) \subset \operatorname{co}(S, \mathcal{J}_S)$. Using the monotonicity of the operator conv and applying [2, Theorem 5.10] and [3, Proposition 3.1] we get

$$S \subset \operatorname{conv}(S) = \operatorname{conv}(E(S)) \subset \operatorname{conv}(\operatorname{co}(S, \mathcal{J}_S)) = \operatorname{co}(S, \mathcal{J}_S)$$

which ends the proof. □

In the proof of Proposition 3.2 we used the well-known fact that for any finite set S in a linear space $\mathbf{L}$ we have $\operatorname{conv}(S) = \operatorname{conv}(E(S))$. As the following example illustrates we do not always have $\operatorname{co}(S, \mathcal{J}_S) = \operatorname{co}(E(S), \mathcal{J}_{E(S)})$.

Example 3.3. Take $S = \{x_1, x_2, x_3, x_4\} \subset \mathbb{R}^2$ and $\mathcal{J}_S$ as in [3, Example 2.7]. In each of the six cases considered in that example and illustrated in Figures 7–12 in [3] we have $E(S) = \{x_1, x_2, x_3\}$ and $\operatorname{co}(E(S), \mathcal{J}_{E(S)}) = \operatorname{conv}(S) \neq \operatorname{co}(S, \mathcal{J}_S)$. Notice that the case presented in Figure 12 provides an evidence that in Proposition 3.2 the assumption that $0 \in I_j$ when $x_j \in S \setminus E(S)$ is essential. □

Although we cannot expect to have always $\operatorname{co}(S, \mathcal{J}_S) = \operatorname{co}(E(S), \mathcal{J}_{E(S)})$, nevertheless for some special family $\mathcal{J}^*_{E(S)}$ we do have the equality

$$\operatorname{co}(S, \mathcal{J}_S) = \operatorname{co}(E(S), \mathcal{J}^*_{E(S)}).$$

This will be shown in the next theorem which can be considered as a version of the celebrated Minkowski-Krein-Milmann's Theorem, cf. [1, 2]. Before presenting this theorem we recall some definitions of concepts used in its proof. Let δ be a nonzero real number and $v \in \mathbf{L}$. The transformation $H_v^\delta : \mathbf{L} \to \mathbf{L}$ defined by

$$H_v^\delta(x) = v + \delta\, x$$

is called a *homothety.* The vector v is called the *center of homothety* and the number δ is called the *ratio of the homothety.* If $\delta > 0$ the homothety is called positive and if $\delta < 0$ the homothety is called negative. The image of $K \subset \mathbf{L}$ under H_v^δ is denoted by $H_v^\delta(K)$ and it is called a *homothet of* K.

Theorem 3.4. *Let $S = \{x_1, ..., x_m\} \subset \mathbf{L}$, $m \geq 2$. Assume that $a_1, \ldots, a_m$ are real numbers such that $1 - \sum_{j=1}^m a_j = \delta \neq 0$. Define the family $\mathcal{J}_S$ as follows*

$$\mathcal{J}_S = \{I_j = [a_j, \delta + a_j] \ : \ j = 1, \ldots, m\}, \quad \textit{when} \quad \delta > 0$$

$$\textit{or} \qquad \mathcal{J}_S = \{I_j = [\delta + a_j, a_j] \ : \ j = 1, \ldots, m\}, \quad \textit{when} \quad \delta < 0.$$

Then there exists a family of intervals $\mathcal{J}^*_{E(S)}$ such that

$$\operatorname{co}(S, \mathcal{J}_S) = \operatorname{co}\left(E(S), \mathcal{J}^*_{E(S)}\right). \tag{23}$$

Proof. Let $S = \{x_1, ..., x_m\} \subset \mathbf{L}$, $m \geq 2$. To prove (23) we plan to show that both the sets are equal homothets. It is easy to check that by virtue of [3, Theorem 6.1] we have

$$\operatorname{co}(S, \mathcal{J}_S) = H_v^\delta(\operatorname{conv}(S)), \tag{24}$$

where

$$v = \sum_{j=1}^{m} a_j x_j \quad \text{and} \quad \delta = 1 - \sum_{j=1}^{m} a_j.$$

We will also show that the set $\operatorname{co}\big(E(S), \mathcal{J}^*_{E(S)}\big)$ is a similar homothet. We may assume that the points of S are labeled in such a way that the first n belong to $E(S)$. Then every point x_j, $j = n+1, \dots, m$, can be written as a convex combination of points from $E(S)$. Hence we have $x_j = \sum_{i=1}^n \alpha_i^j x_i$ for some $\alpha_i^j \in [0,1]$ satisfying $\sum_{i=1}^n \alpha_i^j = 1$. Now the vector v can be expressed in the following form

$$v = \sum_{i=1}^{n} a_i x_i + \sum_{j=n+1}^{m} a_j \Big(\sum_{i=1}^{n} \alpha_i^j x_i \Big) = \sum_{i=1}^{n} \Big(a_i + \sum_{j=n+1}^{m} a_j \alpha_i^j \Big) x_i. \tag{25}$$

Let us put $a_i^* = a_i + \sum\limits_{j=n+1}^{m} a_j \alpha_i^j$ for $i = 1, \dots, n$ and define

$$\mathcal{J}^*_{E(S)} = \{ I_i^* = [a_i^*, \delta + a_i^*] \ : \ i = 1, \dots, n \}, \quad \text{when} \quad \delta > 0$$

or

$$\mathcal{J}^*_{E(S)} = \{ I_i^* = [\delta + a_i^*, a_i^*] \ : \ i = 1, \dots, n \}, \quad \text{when} \quad \delta < 0.$$

From (25) it follows that $v = \sum\limits_{i=1}^{n} a_i^* x_i$. Moreover we have

$$\begin{aligned}
\sum_{i=1}^{n} a_i^* &= \sum_{i=1}^{n} \Big(a_i + \sum_{j=n+1}^{m} a_j \alpha_i^j \Big) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} \sum_{j=n+1}^{m} a_j \alpha_i^j \\
&= \sum_{i=1}^{n} a_i + \sum_{j=n+1}^{m} \sum_{i=1}^{n} a_j \alpha_i^j = \sum_{i=1}^{n} a_i + \sum_{j=n+1}^{m} a_j \sum_{i=1}^{n} \alpha_i^j \\
&= \sum_{i=1}^{n} a_i + \sum_{j=n+1}^{m} a_j = \sum_{i=1}^{m} a_i
\end{aligned}$$

Thus $1 - \sum\limits_{i=1}^{n} a_i^* = 1 - \sum\limits_{i=1}^{n} a_i = \delta \neq 0$. Applying again [3, Theorem 6.1] we have

$$\operatorname{co}(E(S), \mathcal{J}^*_{E(S)}) = H_v^\delta(\operatorname{conv}(E(S)). \tag{26}$$

From the equality $\operatorname{conv}(S) = \operatorname{conv}(E(S))$ we obviously have

$$H_v^\delta(\operatorname{conv}(S)) = H_v^\delta(\operatorname{conv}(E(S))). \tag{27}$$

Comparing (24), (26) and (27) we get

$$\operatorname{co}(S, \mathcal{J}_S) = \operatorname{co}(E(S), \mathcal{J}^*_{E(S)})$$

and the proof is finished. □

The requirements in condition (22) were adapted to define extreme points not only in linear spaces (with respect to the conv operator) but also in other structures, including finite structures, such as: convexity spaces, closure spaces, matroids and antimatroids cf. [1, 8, 12].

In this context it is natural to use the ideas of condition (22) to introduce the notion of an *extreme point with respect to the convex interval hull* in the following definition.

Definition 3.5. A point x_i of a finite set $S = \{x_1, \ldots, x_m\} \subset \mathbf{L}$ is an *extreme point of S with respect to the convex interval hull* $\operatorname{co}(S, \mathcal{J}_S)$ provided

$$x_i \in S \quad \text{but} \quad x_i \notin \operatorname{co}(S_{\hat{i}}, \mathcal{J}_{S_{\hat{i}}}). \qquad \square$$

Now we are ready to introduce the *extreme interval operator* $S \mapsto \operatorname{Eco}(S)$ as

$$\operatorname{Eco}(S) := \{x \in S : x \text{ is an extreme point of } S \text{ with respect to } \operatorname{co}(S, \mathcal{J}_S)\}.$$

The following fact simply follows from the above definition.

Proposition 3.6. *Let x_i be an extreme point of S with respect to $\operatorname{co}(S, \mathcal{J}_S)$ in which every interval I_j contains 0. Then x_i is an extreme points of A with respect to $\operatorname{co}(A, \mathcal{J}_A)$ for any $A \subset S$ containing x_i.*

Proof. If x_i is an extreme point of S with respect to $\operatorname{co}(S, \mathcal{J}_S)$, then

$$x_i \notin \operatorname{co}(S_{\hat{i}}, \mathcal{J}_{S_{\hat{i}}}).$$

If for some $A \subset S$ containing x_i, the vector x_i were not an extreme point of A with respect to $\operatorname{co}(A, \mathcal{J}_A)$, then by Definition 3.5 and Lemma 2.11 we would have

$$x_i \in \operatorname{co}(A, \mathcal{J}_A) \subset \operatorname{co}(S_{\hat{i}}, \mathcal{J}_{S_{\hat{i}}}),$$

a contradiction. $\square$

In the following examples we examine relationships between the sets $E(S)$ and $\operatorname{Eco}(S)$.

Example 3.7. Let $S = \{x_1, x_2, x_3\} \subset \mathbb{R}$, where $x_1 = 0$, $x_2 = 1$, $x_3 = 4$ and let $I_1 = [0.5, 2.5]$, $I_2 = [-1.5, 0.5]$, $I_3 = [0.5, 2.5]$. Obviously, $\operatorname{co}(S_{\hat{1}}, \mathcal{J}_{S_{\hat{1}}}) = [2.5, 8.5]$, $\operatorname{co}(S_{\hat{2}}, \mathcal{J}_{S_{\hat{2}}}) = \{2\}$, $\operatorname{co}(S_{\hat{3}}, \mathcal{J}_{S_{\hat{3}}}) = [-1.5, 0.5]$ and $\operatorname{co}(S, \mathcal{J}_S) = [0.5, 6.5]$.

Thus $$x_i \notin \operatorname{co}(S_{\hat{i}}, \mathcal{J}_{S_{\hat{i}}}) \quad \text{for} \quad i \in \{1, 2, 3\}$$

and therefore $\operatorname{Eco}(S) = S$. Notice that here $x_2 \in \operatorname{Eco}(S) \setminus E(S)$.

Example 3.8. Consider $S = \{x_1, x_2, x_3, x_4\} \subset \mathbb{R}^2$, where $x_1 = (0, 0)$, $x_2 = (1, 0)$, $x_3 = (1, 1)$, $x_4 = (0, 1)$ and let $I_1 = (-\infty, 1]$, $I_2 = I_3 = I_4 = [0, \infty]$. By Theorem 2.8 $\operatorname{co}(S, \mathcal{J}_S)$ is the first quadrant. Clearly, $\operatorname{co}(S_{\hat{1}}, \mathcal{J}_{S_{\hat{1}}})$ is the triangle with vertices x_2, x_3, x_4. Again using Theorem 2.8 we get that $\operatorname{co}(S_{\hat{2}}, \mathcal{J}_{S_{\hat{2}}})$ is the region bounded by the non-negative y-axis and the line $y = x$, $\operatorname{co}(S_{\hat{3}}, \mathcal{J}_{S_{\hat{3}}})$ is the first quadrant and $\operatorname{co}(S_{\hat{4}}, \mathcal{J}_{S_{\hat{4}}})$ is the region bounded by the non-negative x-axis and the line $y = x$. Thus, in this example we have $\operatorname{Eco}(S) = S_{\hat{3}}$ and $x_3 \in E(S) \setminus \operatorname{Eco}(S)$.

Example 3.9. Let $S = \{x_1, x_2, x_3, x_4\} \subset \mathbb{R}^2$, where $x_1 = (-1, 0)$, $x_2 = (1, 0)$, $x_3 = (0, \sqrt{3})$, $x_4 = (0, \frac{\sqrt{3}}{3})$ and let $I_1 = I_2 = I_3 = [0, 1]$ and $I_4 = [\frac{\sqrt{3}-3}{2}, 1]$. The set $\operatorname{co}(S, \mathcal{J}_S)$ is the nonagon shown in Figure 2. Obviously $\operatorname{co}(S_{\hat{4}}, \mathcal{J}_{S_{\hat{4}}}) = \operatorname{conv}(S_{\hat{4}})$ and $x_4 \in \operatorname{co}(S_{\hat{4}}, \mathcal{J}_{S_{\hat{4}}})$. It is easy to check that $x_i \notin \operatorname{co}(S_{\hat{i}}, \mathcal{J}_{S_{\hat{i}}})$ for $i = 1, 2, 3$. Clearly $\operatorname{Eco}(S, \mathcal{J}_S) = E(S) = S_{\hat{4}}$. This example shows that it is possible to have

$$\operatorname{co}(S, \mathcal{J}_S) \neq \operatorname{co}\big(\operatorname{Eco}(S), \mathcal{J}_{\operatorname{Eco}(S)}\big).$$

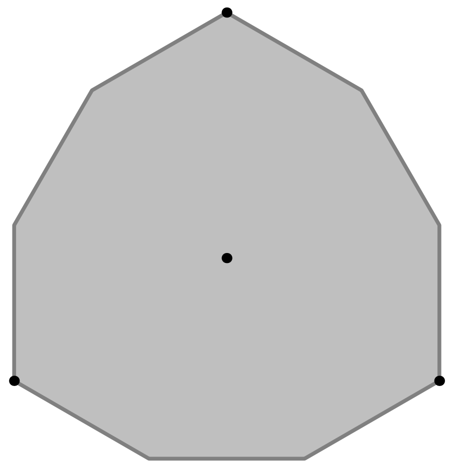

Figure 2: The set $\operatorname{co}(S, \mathcal{J}_S)$ from Example 3.9

Example 3.10. Let $S = \{x_1, x_2, x_3, x_4\} \subset \mathbb{R}^2$, where $x_1 = (-1, -1)$, $x_2 = (1, -1)$, $x_3 = (1, 1)$, $x_4 = (-1, 1)$ and let $I_1 = I_2 = I_3 = I_4 = \mathbb{R}$. Obviously we have $x_i \in \operatorname{co}(S_{\hat{i}}, \mathcal{J}_{S_{\hat{i}}}) = \mathbb{R}^2$ for $i = 1, 2, 3, 4$ and therefore $\operatorname{Eco}(S) = \emptyset$. $\square$

Let S be a finite set in $\mathbf{L}$ and $\mathcal{J}_S$ be a family of intervals. We will say that the convex interval hull $\operatorname{co}(S, \mathcal{J}_S)$ is *absorbing* if for any $A \subset S$ the following implication is true

$$\forall a \in A \quad \big(a \in \operatorname{co}(A \setminus \{a\}, \mathcal{J}_{A\setminus\{a\}}) \implies \operatorname{co}(A \setminus \{a\}, \mathcal{J}_{A\setminus\{a\}}) = \operatorname{co}(A, \mathcal{J}_A)\big).$$

One can easily check that the operators: conv and $\operatorname{co}(S, \mathcal{J}_S)$ in Examples 3.7, 3.8 and 3.10 are absorbing but $\operatorname{co}(S, \mathcal{J}_S)$ in Example 3.9 is not.

Theorem 3.11. *Let* $S = \{x_1, ..., x_m\} \subset \mathbf{L}$ *and* $\mathcal{J}_S = \{I_1, \ldots, I_m\}$ *be a family of intervals such that* $\{0, 1\} \subset I_j$ *for* $j = 1, \ldots, m$. *Assume that* $\operatorname{co}(S, \mathcal{J}_S)$ *is absorbing, then*

$$\operatorname{Eco}(S) = \bigcap \{A \subset S : \operatorname{co}(A, \mathcal{J}_A) = \operatorname{co}(S, \mathcal{J}_S)\}, \tag{28}$$

where $I_j \in \mathcal{J}_A$ *if and only if* $x_j \in A$.

Proof. To show (28) we start with checking if $\operatorname{Eco}(S)$ is contained in the intersection. Obviously the inclusion is true when $\operatorname{Eco}(S) = \emptyset$. If $\operatorname{Eco}(S) \neq \emptyset$ take any $x_k \in \operatorname{Eco}(S)$. If for some $B \subset S$ satisfying $\operatorname{co}(B, \mathcal{J}_B) = \operatorname{co}(S, \mathcal{J}_S)$ we would have $x_k \notin B$ then B would be a subset of $S_{\hat{k}}$. By Lemma 2.11 we would get

$$\operatorname{co}(S, \mathcal{J}_S) = \operatorname{co}(B, \mathcal{J}_B) \subset \operatorname{co}(S_{\hat{k}}, \mathcal{J}_{S_{\hat{k}}}) \subset \operatorname{co}(S, \mathcal{J}_S). \tag{29}$$

Now, by (29) we would have $x_k \in \mathrm{co}(S, \mathcal{J}_S) = \mathrm{co}(S_{\hat{k}}, \mathcal{J}_{S_{\hat{k}}})$. On the other hand, since x_k is an extreme point with respect to $\mathrm{co}(S, \mathcal{J}_S)$ we would have

$$x_k \notin \mathrm{co}(S_{\hat{k}}, \mathcal{J}_{S_{\hat{k}}}),$$

a contradiction.

Denote $\mathcal{A} = \{A \subset S : \mathrm{co}(A, \mathcal{J}_A) = \mathrm{co}(S, \mathcal{J}_S)\}$ and $M = \bigcap \mathcal{A}$. To prove the converse inclusion take any $x_i \in M$. If $x_i \notin \mathrm{Eco}(S)$ then x_i would not be an extreme point with respect to $\mathrm{co}(S, \mathcal{J}_S)$ and therefore $x_i \in \mathrm{co}(S_{\hat{i}}, \mathcal{J}_{S_{\hat{i}}})$. The assumption that $\{0,1\}$ is in every interval $I_j \in \mathcal{J}_S$ ensures that $x_i \in \mathrm{co}(S, \mathcal{J}_S)$. Since $\mathrm{co}(S, \mathcal{J}_S)$ is absorbing we would have $\mathrm{co}(S_{\hat{i}}, \mathcal{J}_{S_{\hat{i}}}) = \mathrm{co}(S, \mathcal{J}_S)$ and therefore $S_{\hat{i}} \in \mathcal{A}$. Because the point x_i belongs to every set in family $\mathcal{A}$ we would have $x_i \in S_{\hat{i}}$, a contradiction. □

Remark 3.12. The inclusion $\mathrm{Eco}(S) \subset \bigcap\{A \subset S : \mathrm{co}(A, \mathcal{J}_A) = \mathrm{co}(S, \mathcal{J}_S)\}$ was proved without using the absorbing assumption of $\mathrm{co}(S, \mathcal{J}_S)$. This assumption was only used to show that the converse inclusion in (28) holds true. Notice that without this assumption equality (28) is not guaranteed. For instance, in Example 3.9 the only subset A of S for which we have $\mathrm{co}(A, \mathcal{J}_A) = \mathrm{co}(S, \mathcal{J}_S)$ is S itself, but we have $\mathrm{Eco}(S) = S_{\hat{4}} \neq S$. Note that the convex interval hull $\mathrm{co}(S, \mathcal{J}_S)$ considered in Example 3.9 is not absorbing and this is the only assumption which is not satisfied there. Thus, the absorbing assumption in Theorem 3.11 is essential and becomes natural in our next theorem. □

As a consequence of Theorem 3.11 we have the following Minkowski-Krein-Milman-type theorem.

Theorem 3.13. *Let $S = \{x_1, ..., x_m\} \subset \mathbf{L}$ and $\mathcal{J}_S = \{I_1, \ldots, I_m\}$ be a family of intervals such that $\{0,1\} \subset I_j$ for $j = 1, \ldots, m$. Assume that $\mathrm{co}(S, \mathcal{J}_S)$ is absorbing and $\mathrm{Eco}(S) \neq \emptyset$, then*

$$\mathrm{co}(S, \mathcal{J}_S) = \mathrm{co}\big(\mathrm{Eco}(S), \mathcal{J}_{\mathrm{Eco}(S)}\big). \tag{30}$$

Proof. Let $\mathcal{A}$ be the family defined in the proof of Theorem 3.11. By Theorem 3.11 we know that $\mathrm{Eco}(S) \subset A$ for every $A \in \mathcal{A}$. We plan to show that $\mathrm{Eco}(S)$ is equal to some set in $\mathcal{A}$ and this will be done in several steps.

Step 1: Of course $\mathcal{A} \neq \emptyset$ because $S \in \mathcal{A}$. If $\mathrm{Eco}(S) = S$ we are done. If not, there exists a point in S, say x_1, which is not an extreme point with respect to $\mathrm{co}(S, \mathcal{J}_S)$. Therefore for x_1 we have

$$x_1 \in \mathrm{co}(S_{\hat{1}}, \mathcal{J}_{S_{\hat{1}}})$$

which with the assumption that $\mathrm{co}(S, \mathcal{J}_S)$ is absorbing gives us

$$\mathrm{co}(S_{\hat{1}}, \mathcal{J}_{S_{\hat{1}}}) = \mathrm{co}(S, \mathcal{J}_S). \tag{31}$$

From (31) and (28) we get

$$S_{\hat{1}} \in \mathcal{A} \quad \text{and} \quad \mathrm{Eco}(S) \subset S_{\hat{1}}. \tag{32}$$

Step 2: If $\mathrm{Eco}(S) = S_{\hat{1}} \in \mathcal{A}$ we are done. If not, in $S_{\hat{1}}$ there is a point, say x_2, such that $x_2 \in S_{\hat{1}} \setminus \mathrm{Eco}(S)$. The same reasoning as was used for x_1 gives

$$x_2 \in \mathrm{co}(S_{\hat{2}}, \mathcal{J}_{S_{\hat{2}}}) \tag{33}$$

and
$$\operatorname{co}(S_{\hat{2}}, \mathcal{J}_{S_{\hat{2}}}) = \operatorname{co}(S, \mathcal{J}_S) \tag{34}$$

and also
$$S_{\hat{2}} \in \mathcal{A} \quad \text{and} \quad \operatorname{Eco}(S) \subset S_{\hat{2}}. \tag{35}$$

Condition (33) allows us to write

$$x_2 = a_1 x_1 + \sum_{i=3}^{m} a_i x_i \in \operatorname{co}(S_{\hat{2}}, \mathcal{J}_{S_{\hat{2}}}),$$

where $a_i \in I_i$, $i \in 1, 3, \ldots, m$ and $a_1 + \sum_{i=3}^{m} a_i = 1$.

By (31) and (34) we have $\operatorname{co}(S_{\hat{1}}, \mathcal{J}_{S_{\hat{1}}}) = \operatorname{co}(S_{\hat{2}}, \mathcal{J}_{S_{\hat{2}}})$. Thus

$$x_2 = a_1 x_1 + \sum_{i=3}^{m} a_i x_i \in \operatorname{co}(S_{\hat{1}}, \mathcal{J}_{S_{\hat{1}}}).$$

Observe that the vector x_2 and the corresponding interval I_2 make no contribution to the linear combination $a_1 x_1 + \sum_{i=3}^{m} a_i x_i$ belonging to $\operatorname{co}(S_{\hat{1}}, \mathcal{J}_{S_{\hat{1}}})$. For that reason this linear combination also belongs to $\operatorname{co}(S_{\hat{1}\hat{2}}, \mathcal{J}_{S_{\hat{1}\hat{2}}})$. Hence

$$x_2 \in \operatorname{co}(S_{\hat{1}\hat{2}}, \mathcal{J}_{S_{\hat{1}\hat{2}}}). \tag{36}$$

From (36) together with the absorbing assumption and the fact that $S_{\hat{1}} \in \mathcal{A}$ we conclude

$$x_2 \in \operatorname{co}(S_{\hat{1}\hat{2}}, \mathcal{J}_{S_{\hat{1}\hat{2}}}) \Longrightarrow \operatorname{co}(S_{\hat{1}\hat{2}}, \mathcal{J}_{S_{\hat{1}\hat{2}}}) = \operatorname{co}(S_{\hat{1}}, \mathcal{J}_{S_{\hat{1}}}) = \operatorname{co}(S, \mathcal{J}_S). \tag{37}$$

Hence $S_{\hat{1}\hat{2}} \in \mathcal{A}$. Now the observations in (32) and (35) together with (28) allow us to summarize Step 2 in the following way

$$\operatorname{Eco}(S) \subset S_{\hat{1}} \cap S_{\hat{2}} = S_{\hat{1}\hat{2}} \in \mathcal{A}.$$

Step 3: If $\operatorname{Eco}(S) = S_{\hat{1}\hat{2}} \in \mathcal{A}$ we are done. If not, in $S_{\hat{1}\hat{2}}$ there is a point, say x_3, such that $x_3 \in S_{\hat{1}\hat{2}} \setminus \operatorname{Eco}(S)$. In a very similar way as in Step 2 we obtain

$$x_3 = c_1 x_1 + c_2 x_2 + \sum_{i=4}^{m} c_i x_i \in \operatorname{co}(S_{\hat{3}}, \mathcal{J}_{S_{\hat{3}}}) = \operatorname{co}(S_{\hat{1}\hat{2}}, \mathcal{J}_{S_{\hat{1}\hat{2}}}) \tag{38}$$

for some numbers $c_i \in I_i$, $i \in 1, 2, 4, \ldots, m$ such that $c_1 + c_2 + \sum_{i=4}^{m} c_i = 1$. The linear combination from (38) belongs to $\operatorname{co}(S_{\hat{1}\hat{2}}, \mathcal{J}_{S_{\hat{1}\hat{2}}})$ and x_3 and I_3 make no contribution to it. Therefore this combination also belongs to $\operatorname{co}(S_{\hat{1}\hat{2}\hat{3}}, \mathcal{J}_{S_{\hat{1}\hat{2}\hat{3}}})$. Acting similarly as in (37) we can conclude Step 3 with

$$\operatorname{Eco}(S) \subset S_{\hat{1}\hat{2}} \cap S_{\hat{3}} = S_{\hat{1}\hat{2}\hat{3}} \in \mathcal{A}.$$

The assumption that $\operatorname{Eco}(S) \neq \emptyset$ guarantees that a continuation of the procedure described in Steps 1–3 leads to a conclusion that $\operatorname{Eco}(S) \in \mathcal{A}$ and proves equality (30). The proof is complete. □

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